A Semi-Decision Procedure for Validity

We now have a semi-decision procedure for validity of a formula:

- Negate the formula
- Transform the result into clausal form
- Generate an appropriate set of ground instances
- Check to see if the set of ground instances is satisfiable (using the obvious variation on propositional resolution).

The problem is, how do we pick the appropriate set of ground instances?

Unification

Suppose we have the clauses:

$$\forall x(\neg p(x)) \qquad p(a) \vee \neg q(b) \qquad \forall y(q(y))$$

What is the ground-resolution that shows that this set is unsatisfiable?

How about for the set:

$$\forall x(p(x)) \qquad \neg p(a) \vee \neg p(b)$$

Or the set:

$$\forall x \forall y\ p(g(x), x, y) \qquad \forall w\ \neg p(w, a, f(w))$$

Or the set:

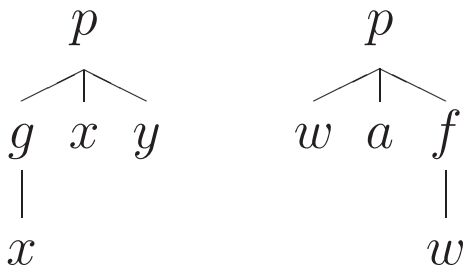
$$\forall x \forall y\ p(g(x), x, y) \qquad \forall w \forall z\ \neg p(w, z, f(w))$$

Unification

Our goal is to determine the ground instantiation *lazilly*, as we are forced into it by resolution.

The solution is *Unification*, which given two terms (or atoms) determines the assignments to the variables in the terms such that the terms could then be made identical. That is, two terms t_1 and t_2 are *unifiable* iff there is a substitution σ such that $t_1\sigma = t_2\sigma$.

The process is naturally described in terms of making the trees of the two terms (or atoms) identical. For example:



Unification

The algorithm can be described roughly as follows:

To unify term t with term s, yielding a unifying *answer substitution*

1. If s is a variable, succeed, with $\sigma = \{s := t\}$.
2. If t is a variable, succeed, with $\sigma = \{t := s\}$.
3. If s is a constant, then:
   (a) If t is the same constant, succeed with $\sigma = \emptyset$.
   (b) Otherwise, fail.
4. If s is a compound term $f(s_1, \ldots, s_n)$, then
   (a) If t is not a compound term $f(t_1, \ldots, t_n)$, fail.
   (b) If t is a compound term $f(t_1, \ldots, t_n)$, then
      i. Set $\sigma_0 = \emptyset$
      ii. For $i \in [1, \ldots, n]$ attempt to unify $s_i\sigma_0 \ldots \sigma_{i-1}$ with $t_i\sigma_0 \ldots \sigma_{i-1}$, yielding σ_i. If any fails, fail.
      iii. Succeed with $\sigma = \sigma_1 \circ \ldots \circ \sigma_n$

Occur's Check

This algorithm is not quite right.

In defining the algorithm we must be careful of one thing. Consider an attempt to resolve the set:

$$\forall x\ p(x, f(x)) \qquad \forall y\ \neg p(y, y)$$

Therefore, the first two steps must be modified to read:

1. If s is a variable, then:
   (a) if $s \notin FV(t)$, succeed, with $\sigma = \{s := t\}$,
   (b) otherwise, fail.
2. If t is a variable, then:
   (a) if $t \notin FV(s)$, succeed, with $\sigma = \{t := s\}$,
   (b) otherwise, fail.

This extra step is called the *occurs check*.

Mr. MGU

It can be shown that for any two unifiable terms, there exists a substitution that makes the fewest possible substitutions for variables of the two terms. This unifying substitution is called the *Most General Unifier*, or *MGU*.

More precisely, if σ is the MGU of terms t_1 and t_2, then for any other unifier σ' there is a substitution σ'' such that $\sigma' = \sigma \circ \sigma''$.

We can further prove that for any two terms, the given unification algorithm always produces the most general unifier if the terms are unifiable.

Unification, By The Book

Definition: (4.3.2) A set of term equations is in *solved form* if:

- All equations are of the form $x = t$
- Each variable x that appears on the left-hand side of an equation does not appear elsewhere in the set.

Unification, By The Book

The unification algorithm can be described in terms of sets of term equations as follows:

To unify the terms s and t, begin with the set of term equations $\{s = t\}$. Perform the following transformations on the set of term equations as long as any one of them is applicable:

1. Replace any equation of the form $t = x$ where t is not a variable, with $x = t$.
2. Erase any equation of the form $x = x$ or $c = c$
3. If there is an equation of the form $s = t$ where neither s nor t is a variable:
   (a) If s is a constant c, then if t is not also the constant c, terminate the algorithm, indicating failure.
   (b) If s is a term of the form $f(s_1, \ldots, s_n)$ then if t is not a term of the form $f(t_1, \ldots, t_n)$, terminate the algorithm, indicating failure. Otherwise, remove the equation $s = t$ and replace it with the n equations $s_1 = t_1$, $\ldots$, $s_n = t_n$.
   (c) If there is an equation of the form $x = t$, then if x occurs in t, terminate, indicating failure. Otherwise, replace all other occurences of the variable x in the set with the term t.

Resolution by the Book

The *General Resolution Rule* is stated as follows:

> Let C_1 and C_2 be clauses with no variables in common. Let $l_1 \in C_1$ and $l_2 \in C_2$ be literals such that l_1 and l_2^c can be unified by an mgu σ. C_1 and C_2 are said to be *clashing* clauses and to *clash* on the literals l_1 and l_2. C, the resolvent of C_1 and C_2 is the clause:
>
> $$Res(C_1, C_2) = (C_1\sigma - \{l_1\sigma\}) \cup (C_2\sigma - \{l_2\sigma\})$$

For example:

$p(f(x), g(y)) \vee q(x, y) \quad \neg p(f(f(a)), g(z)) \vee q(f(a), g(z))$

Standardizing Apart

The restriction that the clauses have disjoint sets of variables is crucial to ensure completeness. If we did not use this restricition, which may require doing an α-conversion on the clauses before resolving, we would not be able to resolve the clauses:

$$p(f(x)) \qquad \neg p(x)$$

due to the occurs check. Yet these represent independent clauses:

$$\forall x(p(f(x))) \qquad \forall x(\neg p(x))$$

which are mutually unsatisfiable.

The process of assigning new variables to the clauses is called *standardizing apart*, or, appplying a *separating pair of substitutions*.

The General Resolution Rule is Very General

While it is not explicitly stated in the discussion of the resolution rule in BenAri, it is important to recognize that the clause $(C_i\sigma - \{l_i\sigma\})$ may be more than one literal smaller than the clause C_i. Consider, for example:

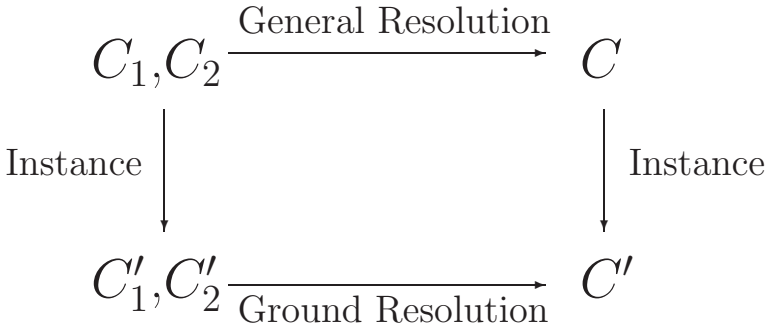
$$\neg p(x, a) \vee \neg p(x, y) \vee \neg p(y, x) \qquad p(z, f(z)) \vee p(z, a)$$

The Lifting Lemma

The completeness of General Resolution is, effectively, a corollary of the following theorem:

Theorem (4.4.1 – Lifting Lemma)**:** Let C_1' and C_2' be ground instances of the clauses C_1 and C_2, respectively. Let C' be a (ground) resolvent of C_1' and C_2', obtained by resolving on the literal l'. Then there is a (general) resolvent, C, of C_1 and C_2, such that C' is a ground instance of C.

This is captured in the following figure:



General Resolution Procedure

The *General Resolution Procedure* is given as:

> Let S be a set of clauses and define $S_0 = S$. Assume that S_i has been constructed. *Choose* clashing clauses $C_1, C_2 \in S_i$ and let $C = Res(C_1, C_2)$. If $C = \square$ then terminate the procedure – S is unsatisfiable. Otherwise, construct $S_{i+1} = S_i \cup \{C\}$. If $S_{i+1} = S_i$ for all pairs of clashing clauses, terminate the procedure – S is satisfiable.