Computer Science 131, Fall 2001
Sample Solution for Assignment 4

Lemma 1 (Inversion Extension)

1. *If* $\Gamma \vdash \langle e_1, e_2 \rangle : t$ *then* $t = t_1 \times t_2$ *for some types* t_1 *and* t_2, *where* $\Gamma \vdash e_1 : t_1$ *and* $\Gamma \vdash e_2 : t_2$.

2. *If* $\Gamma \vdash \texttt{fst}\ e : t$ *then* $\Gamma \vdash e : t \times t_2$ *for some type* t_2.

3. *If* $\Gamma \vdash \texttt{snd}\ e : t$ *then* $\Gamma \vdash e : t_1 \times t$ *for some type* t_1.

Lemma 2 (Canonical Forms Extension)

1. *If* $\vdash v : t_1 \times t_2$ *then* v *is an pair of the form* $\langle v_1, v_2 \rangle$.

Proposition 3 (Type Preservation)
If $\vdash e : t$ *and* $e \rightarrow e'$ *then* $\vdash e' : t$.

Proof: By induction on the proof that $e \rightarrow e'$.

- Case: Rule 29. Then $e = \langle e_1, e_2 \rangle$. By Inversion, $\vdash e_1 : t_1$ and $\vdash e_2 : t_2$ where $t = t_1 \times t_2$. By the inductive hypothesis applied to $e_1 \rightarrow e_1'$, we have $\vdash e_1' : t_1$. Therefore $\vdash \langle e_1', e_2 \rangle : t_1 \times t_2$ as required.

- Case: Rule 30. Then $e = \langle v_1, e_2 \rangle$. By Inversion, $\vdash v_1 : t_1$ and $\vdash e_2 : t_2$ where $t = t_1 \times t_2$. By the inductive hypothesis applied to $e_2 \rightarrow e_2'$, we have $\vdash e_2' : t_2$. Therefore $\vdash \langle v_1, e_2' \rangle : t_1 \times t_2$ as required.

- Case: Rule 31. Then $e = \texttt{fst}\ e_1$ and $e' = \texttt{fst}\ e_1'$ where $e_1 \rightarrow e_1'$. By inversion, $\vdash e_1 : t \times t_2$ for some type t_2. By the inductive hypothesis, $\vdash e_1' : t \times t_2$. Thus $\vdash \texttt{fst}\ e_1' : t$ as required.

- Case: Rule 32. Then $e = \texttt{fst}\ \langle v_1, v_2 \rangle$ and $e' = v_1$. By inversion, $\vdash \langle v_1, v_2 \rangle : t \times t_2$ for some type t_2. By inversion again, $\vdash v_1 : t$, as required.

- Case: Rules 33 and 34. Exactly analogous to the two previous cases.

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Proposition 4 (Progress)
If $\vdash e : t$ *then either* e *is a value or there exists* e' *such that* $e \rightarrow e'$.

Proof: By induction on the proof of $\vdash e : t$, and cases on the last rule used.

- Case: Rule 26. Then $e = \langle e_1, e_2 \rangle$ and $t = t_1 \times t_2$ and there are sub-proofs $\vdash e_1 : t_1$ and $\vdash e_2 : t_2$. There are three subcases to consider.
  - Subcase: e_1 is not a value. By the inductive hypothesis, there exists e_1' such that $e_1 \rightarrow e_1'$. Thus by Rule 29 we have $\langle e_1, e_2 \rangle \rightarrow \langle e_1', e_2 \rangle$.
  - Subcase: e_1 is a value, but e_2 is not. By the inductive hypothesis there exists e_2' such that $e_2 \rightarrow e_2'$. By Rule 30 we have $\langle e_1, e_2 \rangle \rightarrow \langle e_1, e_2' \rangle$.
  - Subcase: e_1 and e_2 are both values. Then the pair $\langle e_1, e_2 \rangle$ is also a value.
- Case: Rule 27. Then $e = \texttt{fst}\ e_1$ and and there is a sub-proof $\vdash e_1 : t \times t_2$ for some type t_2. There are two subcases to consider:
  - Subcase: e_1 is not a value. By the inductive hypothesis there exists e_1' such that $e_1 \rightarrow e_1'$. Thus $e \rightarrow \texttt{fst}\ e_1'$.
  - Subcase: e_1 is a value. By the Canonical Forms lemma, e_1 must be of the form $\langle v_1, v_2 \rangle$ for some values v_1 and v_2. Thus by Rule 32 we have $e \rightarrow v_1$.
- Case: Rule 28. Exactly analogous to the previous case.

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