# CS140: Algorithms

## Spring 2001

### Solutions to Assignment 4

1. Complete the last step in the analysis of Randomized Select by showing that $f(n) = O(n)$ where

   (a) $f(1) = c$

   (b) For $n > 1$, $f(n) \leq cn + \frac{2}{n}\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} f(i-1)$.

   Ans. We will show that $f(n) \leq 4cn$ for all $n \geq 1$.

   For $n = 1$, $f(n) = c < 4c$.

   Now suppose that $n > 1$ and that the claim holds for $f(i) < n$. Then

$$
\begin{array}{rcll}
f(n) & \leq & cn + \frac{2}{n}\sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} f(i-1) & \text{definition} \\
 & = & cn + \frac{2}{n}\sum_{i=\lceil \frac{n+1}{2} \rceil - 1}^{n-1} f(i) & \text{change of variable} \\
 & \leq & cn + \frac{2}{n}\sum_{i=\lceil \frac{n+1}{2} \rceil - 1}^{n-1} 4c \cdot i & \text{using inductive hypothesis} \\
 & = & cn + \frac{2}{n}(\sum_{i=1}^{n-1} 4c \cdot i \ - \ \sum_{i=1}^{\lceil \frac{n+1}{2} \rceil - 2} 4c \cdot i) & \text{rewriting the summation} \\
 & = & cn + \frac{2}{n}(4c\frac{n(n-1)}{2} \ - \ 4c\frac{(\lceil \frac{n+1}{2} \rceil - 2)(\lceil \frac{n+1}{2} \rceil - 1)}{2}) & \text{solving the arithmetic series} \\
 & = & cn + 4cn(\frac{n-1}{n} \ - \ \frac{1}{n^2}(\lceil \frac{n+1}{2} \rceil - 2)(\lceil \frac{n+1}{2} \rceil - 1)) & \text{simplifying} \\
 & \leq & cn + 4cn(\frac{3}{4}) & \text{bounding the ceiling function} \\
 & = & 4c &
\end{array}
$$

2. Complete the last step in the analysis of the Select algorithm that uses median-of-medians pivots by showing that $f(n) = O(n)$ where

   (a) $f(1) = c$

   (b) For $n > 1$, $f(n) \leq cn + f(n/5) + f(3n/4)$

   Ans. We will prove that $f(n) \leq 20c$ for all $n \geq 1$.

   For $n = 1$, $f(n) = c < 20c$.

   Now suppose that $n > 1$ and that the claim holds for $f(i)$ where $i < n$. Then

$$
\begin{array}{rcll}
f(n) & \leq & cn + f(n/5) + f(3n/4) & \text{definition} \\
 & \leq & cn + 20cn/5 + 20c(3n/4) & \text{inductive hypothesis} \\
 & = & 20cn &
\end{array}
$$

3. A pair of fair die are tossed. Let $X$ be the sum of the values that appear.

   (a) What is the sample space of $X$?
   Ans. $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

   (b) What are the probabilities of the elementary events in the sample space?
   Ans. $P(X = 2) = P(X = 12) = 1/36$, $P(X = 3) = P(X = 11) = 2/36$, $P(X = 4) = P(X = 10) = 3/36$, $P(X = 5) = P(X = 9) = 4/36$, $P(X = 6) = P(X = 8) = 5/36$, $P(X = 7) = 6/36$

   (c) What is $E[X]$?
   Ans. $E[X] = 7$

   (d) What is $E[X^2]$?
   Ans. $E[X^2] = 54.75$