A Single-Exponential Upper Bound for Finding Shortest Paths in Three Dimensions

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Abstract. We derive a single-exponential time upper bound for finding the shortest path between two points in 3-dimensional Euclidean space with (nonnecessarily convex) polyhedral obstacles. Prior to this work, the best known algorithm required double-exponential time. Given that the problem is known to be PSPACE-hard, the bound we present is essentially the best (in the worst-case sense) that can reasonably be expected.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

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1. *Introduction*

We consider the problem of finding a minimum length path between two points in Euclidean space that avoids a set of (not necessarily convex) polyhedral obstacles. Because this problem is known to be PSPACE complete [Canney and Reif 1987], the worst-case complexity of this problem is in some-sense a theoretical issue. Here, we employ the theory of real closed fields to derive a single exponential time upper bound for finding the shortest path between two points in 3-dimensional Euclidean space with (nonnecessarily

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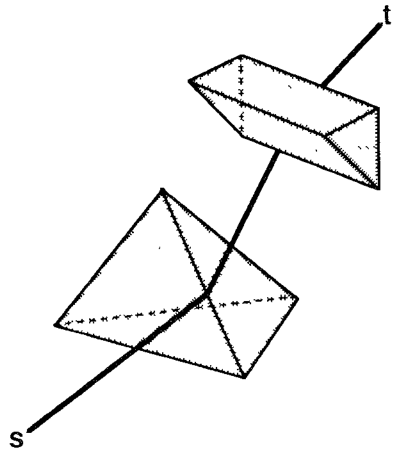

FIG. 1. Shortest paths may not bend at obstacle vertices.

convex) polyhedral obstacles. Prior to this work, the best-known algorithm required double-exponential time [Sharir and Schorr 1984].

As depicted in Figure 1, one of the inherent difficulties with computing shortest paths in three-dimensional spaces is that, although shortest paths are sequences of line segments that start and end on obstacle edges, they do not necessarily pass through obstacle vertices. Furthermore, the length of a minimal length path, as well as the coordinates through which it passes, may not be rational numbers (e.g., Bajaj [1985] shows that the problem is not, in general, solvable by radicals). At best, one can only output a minimal path to within a given accuracy.

Sharir and Schorr [1984] present a doubly exponential time algorithm to test whether a given length is achievable. Papadimitriou [1985] presents a fully polynomial approximation scheme for computing an approximate rational path,[1] but this does not provide a test as to whether a given rational or algebraic length is achievable; Clarkson [1987] also considers approximation algorithms.

It should be noted that efficient algorithms are known for some special cases. Both Franklin and Akman [1984] and Sharir and Schorr [1984] give a polynomial-time algorithm for finding the shortest path on the surface of a convex polyhedron between two points. O'Rourke et al. [1984] generalize the work of Sharir and Schorr [1984] to obtain an $O(n^5)$ algorithm for minimal movement on the surface of a nonconvex polyhedron, and Mitchell et al. [1986] improve their bound to $O(n^2 \log(n))$. Baltsan and Sharir [1988] present a polynomial-time algorithm for computing a shortest path between points on two convex polyhedra, and Sharir and Baltsan [1986] present a polynomial-time algorithm for the shortest path between two points among a number of convex polyhedra. Bajaj [1986] presents a parallel numerical iterative method for the case where the order of encountered obstacles is known. Franklin et al. [1984] consider generalizations of the Voronoi diagram that may be useful for algorithms relating to shortest path problems.

[1] That is, an algorithm is presented that finds a path of length at most $1 + \epsilon$ times the length of a shortest path, and works in time polynomial in n (the number of obstacle edges), L (the maximum number of bits used to describe an obstacle vertex coordinate), and $1/\epsilon$.

We present an $n^{k^{O(1)}}$ time algorithm for finding the shortest path between two points (that avoids arbitrary polyhedral obstacles), where n denotes the *size* of the obstacle space (number of obstacle edges) and k denotes the number of *islands* (number of convex connected components). This yields polynomial time when k is $O(1)$ and at most single-exponential time in general. Our algorithm works as follows: First, we show how to express a minimal length path as a formula; second, we show how to rewrite this formula so that it is polynomial in length; third, we show that the formula need only have $O(\log(k))$ variables. Then, we can employ previously known algorithms for the theory of real closed fields.

2. *Preliminaries*

Our model of computation for our 3-space algorithm is the standard log-cost RAM.[2]

Definition 2.1. A three-dimensional *obstacle space* consists of a set of disjoint polyhedra. An *island* is a connected convex subset of the obstacle space. That is, for any two points contained in the interior of the island, a shortest path between these two points is strictly contained in the interior of the island.

Note that a set of i disconnected convex polyhedra forms a set of i islands, however, a single nonconvex polyhedra constitutes more than one island. For example, Figure 2 constitutes three islands.

We use the following notation:

—n denotes the number of obstacle edges and k the minimum number of islands that can represent the space. Note that k can be computed in exponential time and polynomial space by simply enumerating possible partitions of the obstacle space.[3]

—$E = \{e_1 \cdots e_n\}$ denotes the set of obstacle edges; for convenience, we assume that obstacle edges are directed, so that we can talk about traversing an obstacle edge from its "start point" to its "end point."

—If edge $e = (u, v)$ has length l, then for $0 \leq \delta \leq 1$, let $e(\delta)$ denote the point on e of distance δl from u (i.e., $e(0) = u$ and $e(1) = v$).

—$s = e_1(0)$ denotes the source vertex and $t = e_n(0)$ denotes the destination vertex.

[2] See, for example, Aho et al. [1974].

[3] Determining a minimum partition into convex polygons is NP-complete even for two dimensions [Johnson 1982; Lingas 1982; Keil 1983] and even when the convex polygons are restricted to be convex quadrilaterals or trapezoids [Lubiw 1985; Asano et al. 1986]. Hence, if we wish to "optimize" the size of the exponent (which depends on k) in the running time of our algorithm, computing k in exponential time and polynomial space is in some sense the best that can be hoped for. However, exponential time will follow for any nonoptimal partition that has a polynomial number of edges (such partitions can easily be computed in polynomial time). In any case, any way of determining k that takes exponential or less time suffices since it is within the running time of our algorithm.

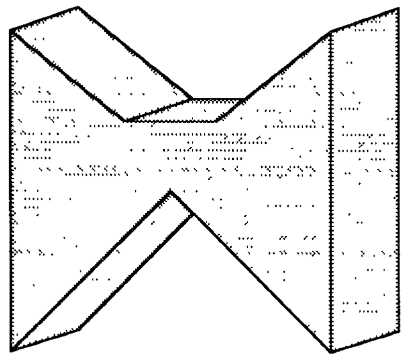

FIG. 2. This object constitutes three islands.

Without loss of generality, we always assume that s and t are vertices of some island. That is, if s and/or t are not on the surface of some island, islands consisting of single points can be created for them (such an island can be thought of as single edge of length 0). If s and/or t are on island surfaces but are not vertices, at most four new obstacle edges can be added to make s and/or t a vertex. For example, if s is in the interior of some face F, then pass two line segments through s to divide F into four faces. Note that these new four faces will be co-planar, but this does not effect our construction.

Definition 2.2. A *direct* path in a three-dimensional obstacle space is a straight-line segment that does not intersect any island except at its endpoints. A *contact* path is a one-dimensional curve that lies entirely on the surface of an island and has the property that is a minimal length path between its endpoints. A *fundamental* path is one that consists of a direct path followed by a contact path.

Fact 2.1 [Sharir and Schorr 1984]. A contact path between two points a and b that lie on (possibly different) edges of an island consisting of $O(m)$ edges consists of a sequence of at most $O(m)$ straight-line segments such that the endpoints of these line segments lie on the edges of the island. In addition, this contact path can be computed in time polynomial in m.

Definition 2.3. A *normal* path is a sequence of (at most k) fundamental paths that intersect each island in at most one contact path.[4]

LEMMA 2.1. *In a three-dimensional obstacle space with k islands, if a shortest path between two points s and t has length l, then there is a normal path between s and t of length l.*

PROOF. This follows from the observation that the intersection of any minimal length path with an island must be a connected set.

An *open formula* $F(x_1 \cdots x_m)$ in the theory of real closed fields consists of a logical expression containing conjunctions, disjunctions, negations, and inequalities between rational polynomials in the variables $x_1 \cdots x_m$. A (partially quantified) formula in this theory is of the form $Q_1 x_{i_1} \cdots Q_r x_{i_r} F(x_1 \cdots x_m)$ where the Q_i denote quantifiers and $r \leq m$; its *degree* is the maximum degree of any polynomial within the formula. Our approach is to show that for a given

[4] This definition is motivated by ideas from Reif and Sharir [1984].

length l, we can describe by a formula (with relatively short length and small number of quantifiers) the paths of length l between the source and destination. Once this has been done, we can make use of the following facts:

Fact 2.2 [Collins 1975].[5] Given a formula in the theory of real closed fields of length l, degree d, and v variables, satisfiability can be tested in time $(dl)^{2^{O(v)}}$.

Fact 2.3 [Ben-Or et al. 1984]. Given a formula in the theory of real closed fields of length l, degree d, and v variables, satisfiability can be tested using space $(dl)^{O(v)}$.

3. *A Single-Exponential Bound*

THEOREM 3.1. *Given two points s and t in a three-dimensional obstacle space and a length l, in $n^{k^{O(1)}}$ time, it is possible to determine whether there is a path between s and t of length l.*

PROOF. We first define the following predicates:

$D(i_1, i_2, \delta_1, \delta_2, l)$ is true if and only if there exists a direct path from $e_{i_1}(\delta_1)$ to $e_{i_2}(\delta_2)$ of length l.

$C(i_1, i_2, \delta_1, \delta_2, l)$ is true if and only if there exists a contact path from $e_{i_1}(\delta_1)$ to $e_{i_2}(\delta_2)$ of length l.

$N(i_1, i_2, \delta_1, \delta_2, l)$ is true if and only if there exists a normal path from $e_{i_1}(\delta_1)$ to $e_{i_2}(\delta_2)$ of length l.

A polynomial length formula for D can be constructed by checking for visibility and a polynomial length formula for C follows from Fact 6.1. A formula for N can be constructed as follows:

$$\begin{aligned} N^{(1)}&(i_1, i_2, \delta_1, \delta_2, l) \\ &= \exists i_3 \delta_3 l_1 l_2 ((l = l_1 + l_2) \wedge D(i_1, i_3, \delta_1, \delta_3, l_1) \wedge C(i_3, i_2, \delta_3, \delta_2, l_2)) \\ &\quad \times N^{(2j)}(i_1, i_2, \delta_1, \delta_2, l) \\ &= \exists i_3 \delta_3 l_1 l_2 \big((l = l_1 + l_2) \wedge N^{(j)}(i_1, i_3, \delta_1, \delta_3, l_1) \wedge N^{(j)}(i_3, i_2, \delta_3, \delta_2, l_2)\big) \\ N &= N^{(2^{\lceil \log_2(k) \rceil})} \end{aligned}$$

As written above, the formula for N is not polynomial in length, owing to subformula duplication in the definition of $N^{(2j)}$. However, this duplication can be eliminated by the transformation that replaces the expression

$$f(\bar{a}) \wedge f(\bar{b}r)$$

by the expression

$$\forall \bar{x} r\big((\bar{x} = \bar{a} \vee x = \bar{b}) \to f(\bar{x})$$

($\bar{a}$, $\bar{b}$, and $\bar{x}$ denote vectors of five variables). It is easy to check that with this modification the above definition of $N^{(2i)}$ yields a formula for N that has polynomial length, constant degree, and $O(\log(k))$ variables. Hence, the theorem follows from Fact 2.2.

[5]For slightly improved bounds, the reader can refer to Chistov and Grigor'ev [1985].

COROLLARY 3.1a. *Within the time bounds stated by Theorem* 3.1, *the length of a path between s and t can be computed to within a polynomial number of bits of accuracy.*

PROOF. Given the test procedure provided by Theorem 3.1, these bits can be determined with binary search.

COROLLARY 3.1b. *Given two points s and t in a three-dimensional obstacle space and a length l, in* $n^{O(\log(k))}$ *space it is possible to determine whether there is a path between s and t of length l.*

PROOF. Use Fact 2.3 instead of Fact 2.2 in the proof of Theorem 3.1.

Within the bounds stated by Theorem 3.1, the length of a path can be computed to within a polynomial number of bits of accuracy. This may be no more accurate than what can be achieved by approximation algorithms such as those mentioned in the introduction. However, a key advantage of the construction of Theorem 3.1 is that it is always possible to test whether a given rational or algebraic length is achievable, even if more than a polynomial number of bits of accuracy are needed to define the outcome of the test.

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