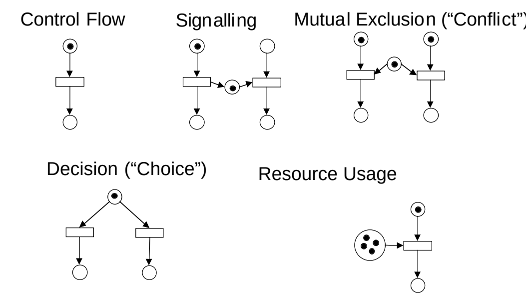
Modeling Tools

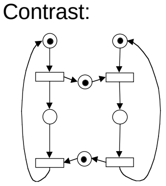
- Why prefer one model over another?
  - Clarity
  - Expressiveness/Completeness
  - Analytic possibilities
  - Implementation possibilities

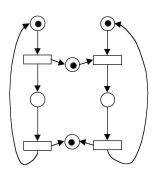
Review of Basic Petri Net Constructs



Boundedness of Petri Nets

- Def: A Petri Net is ***bounded*** **wrt** a specific initial state if the set of states reachable from the initial state is finite.
  Contrast:





Boundedness

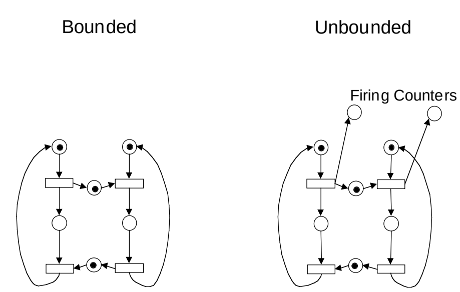
- In general, boundedness is a “good thing”.
- It is essential if the system is to be realized with finite memory.
- It allows the system to be analyzed as a finite-state machine.
- However, a type of unboundedness can be useful for mathematical *analysis*, e.g. in the form of ***firing counters***.

When Boundedness is Undesirable

- The system being modeled might **not** be finite-state, *e.g.*:
  - Producer/Consumer problem with no limit on production.
  - Artifacts for mathematical *analysis*, e.g. in the form of ***firing counters***.

Firing Counters

should be unbounded as a ***necessary*** condition to be free of deadlock



Boundedness of Individual Places

- Def: A set of *places* (circular nodes) in a Petri net is **simultaneously unbounded** w.r.t. an initial state if
  - ($\forall n \in \{0, 1, 2, 3, \ldots\}$) there is a reachable state in which each place has $\geq$ n tokens.

Theorem (Karp & Miller, 1966)

- There is an algorithm for deciding whether any given set of places in a Petri net is simultaneously unbounded.
- Note: Karp & Miller did not state this result in terms of Petri nets; they used a different, but more-or-less equivalent model called ***Vector Addition Systems***.

Vector Addition Systems

- Consider representing the states of a Petri net as vectors:
  - The vector components are identified with places.
  - Firing a transition has the effect of adding a vector (which may have negative components) to the state (which never has negative components).
  - The enabling rule requires that state components *stay* non-negative.

Monotonicity: Key Insight for Vector Addition Systems & Petri Nets

- This insight is what makes certain decidability results possible, and is also what creates certain computational limitations on these particular models.
- Let $q_0 -t_1 \rightarrow q_1 -t_2 \rightarrow \ldots q_{n-1} -t_n \rightarrow q_n$ be a firing sequence leading from a state q_0. Let $q'_0 \geq q_0$. Then there is also a firing sequence
  $q'_0 -t_1 \rightarrow q'_1 -t_2 \rightarrow \ldots q'_{n-1} -t_n \rightarrow q'_n$

  (where $(\forall i)\ q'_i = q_i + (q'_0 - q_0)$).

Monotonicity

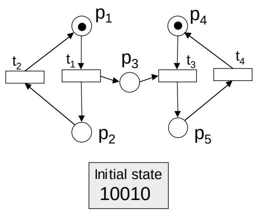
- In particular, if
  $q_0 -t_1 \rightarrow q_1 -t_2 \rightarrow \ldots q_{n-1} -t_n \rightarrow q_n$
  where $q_n > q_0$
  (strict inequality in one or more components),

  then there exist sets of reachable states in which those components are without bound.

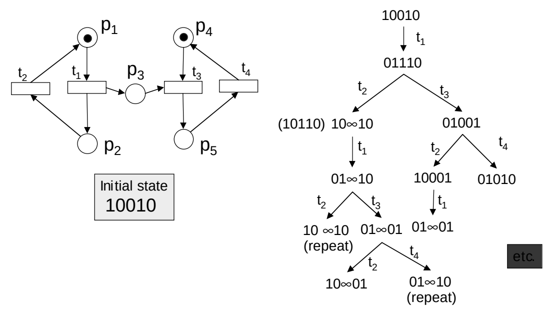
Reachability-Tree Algorithm

- Construct a tree with the initial state as root.
- Construct successive nodes for each firable transition, as if constructing a state diagram.
- Whenever a node is added that has a predecessor which is pointwise $\leq$ this node, set to ∞ any place that is < in the predecessor. If the result is just a repeat, that branch ends.
- This process will terminate. Nodes having sets of places with ∞ indicate sets of places that are simultaneously unbounded.

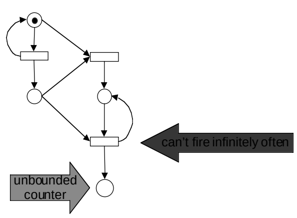
Reachability-Tree Algorithm



Reachability-Tree Algorithm



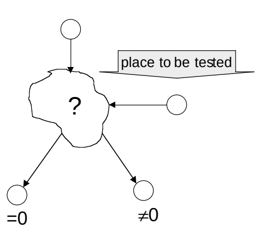
Unboundedness of a Counter for a Transition is a **Necessary, but not Sufficient**, Condition for the Transition to have an Infinite Firing Sequence



Monotonicity Revisited

- Monotonicity is what makes the reachability tree possible.
- Monotonicity also points out a modeling limitation of Petri nets:
  - For unbounded Petri nets, it is not possible to devise a “0-testing structure”.

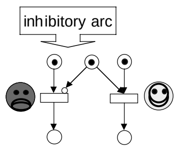
Hypothetical 0-testing structure



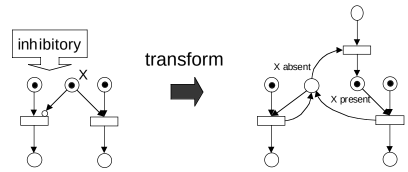
Generalizing Petri Nets

- Adding inhibitory arcs:
  - For finite-state systems: ok, can simulate without inhibitory arcs anyway.
  - For unbounded systems: can destroy essential decidability properties (now can simulate a Turing machine)
- “Colored” tokens (see Kurt Jensen, 3 vols., Springer, 1997)
- Program variables
- Enabling predicates on transitions

Inhibition

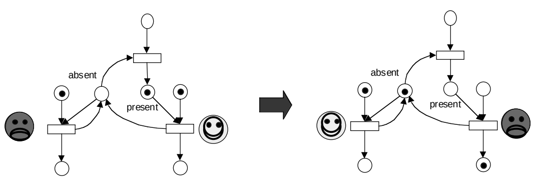


Simulating Inhibitory Arcs



This transformation only works if the place X in question is bounded by 1.

Simulating Inhibitory Arcs



State-to-State Reachability Problem

- This is the question of deciding whether one state can be reached from another.
  - This problem lay open for about 20 years, and was answered in the affirmative by Ernst Mayr.
  - The algorithm is complex.
  - Richard Lipton earlier showed that this problem is **exponential-space** hard.