

Harvey Mudd College  
Computer Science 80  
Logic for Computer Science  
Spring Semester 2001

Assignment #2 – Propositional Logic, Syntax and Semantics  
Due 11:00am, Wednesday February 7, 2001

1. The proofs in this problem have as an immediate corollary that no proper prefix of a well-formed propositional formula is itself a well-formed propositional formula:

Given a set of propositional letters,  $P$ , let  $\Sigma$  be the alphabet of the propositional language over  $P$  as defined in class. Define the function  $K : \Sigma \rightarrow \mathcal{Z}$  as follows:

- $K( ( ) ) = -2, K( ) = 2$
- $K( p ) = 1$ , for all  $p \in P \cup \{ \perp, \top \}$ .
- $K( \neg ) = 0$
- $K( \wedge ) = K( \vee ) = K( \Rightarrow ) = K( \Leftarrow ) = K( \equiv ) = -1$

Further define the function  $K' : \Sigma^* \rightarrow \mathcal{Z}$  as  $K'(u_1 \dots u_n) = K(u_1) + \dots + K(u_n)$ .

- (a) Prove that for any  $\Phi \in PROP$ ,  $K'(\Phi) = 1$ .  
(Hint: Proceed by complete induction on the length of the formula  $\Phi$ .)
- (b) (extra credit) Prove that for any  $w \in \Sigma^*$  such that there is a  $\Phi \in PROP$  such that  $w$  is a proper prefix of  $\Phi$ ,  $K'(w) \neq 1$ .  
(Hint: Though it may seem counter intuitive, it is easiest to proceed again by complete induction on the length of  $\Phi$ , ignoring the length of  $w$ , and to prove a slightly stronger property, that  $K'(w) < 1$ .)
- (c) Conclude that if  $\Phi, \Psi \in PROP$ ,  $\Phi$  then is not a proper prefix of  $\Psi$ .
2. For each of the following formulas, give the formation tree of the formula and a truth-table proof that the formula is a valid:

- (a)  $((a \wedge b) \Rightarrow c) \equiv (a \Rightarrow (b \Rightarrow c))$
- (b)  $((a \Rightarrow b) \Rightarrow ((c \Rightarrow a) \Rightarrow (c \Rightarrow b)))$
- (c)  $((a \Rightarrow b) \Rightarrow a) \Rightarrow a$

(Note: While I think you can probably make sense of why the first two formulas are valid by thinking about the meaning of the operators, that is probably not true of the third. That formula, which is called Pierce's Law, seems to contradict our notion of the meaning of implication. In fact, it is not a valid formula of intuitionistic logic, in which implication behaves much more according to our expectations.)

3. Prove the following theorem from the lectures:

**Theorem (2.5.13):** Given formula  $\Psi \in PROP$  and a set of formulas  $U = \{\Phi_1, \dots, \Phi_n\} \subseteq PROP$ , if  $U \models \Psi$  and there is an  $1 \leq i \leq n$  such that  $\Phi_i$  is valid, then  $U - \{\Phi_i\} \models \Psi$ .

4. Prove the following equivalences by appeal to the equivalences given in class (and in Figure 2.9 of the text). That is, give a series of rewritings of the formulas on the left and right until they arrive at the same point. (Or just go from left to right, if that seems more appropriate).

(a)  $((a \wedge b) \Rightarrow c) \leftrightarrow (a \Rightarrow (b \Rightarrow c))$

(b)  $((a \Rightarrow b) \Rightarrow a) \leftrightarrow \top$

5. Prove that the set  $\{\downarrow\}$  (NOR) is logically complete by giving formulas equivalent to each of the following that use only this one operator:

(a)  $(\neg a)$

(b)  $(a \wedge b)$

(c)  $(a \vee b)$

(d)  $(a \Rightarrow b)$

(e)  $(a \equiv b)$