

Harvey Mudd College
 Computer Science 80
 Logic for Computer Science
 Spring Semester 2001

Assignment #3 – Propositional Logic: Natural Deduction
Sample Solution

1. Give Natural Deduction proofs of each of the following formulas (brackets are used in place of parentheses in some formulas to make the structure clearer):

(a) $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

Note that in the answers to this problem, any leaf with a superscripted number attached is a discharged assumption, and should be assumed to be crossed out. (I could not figure out how to get L^AT_EX to cross out large expressions.)

$$\frac{\frac{\frac{q \wedge p^2}{q} \wedge_E \quad \frac{\frac{q \wedge p^2}{p} \wedge_E \quad p \Rightarrow (q \Rightarrow r)^1}{q \Rightarrow r} \Rightarrow_E}{r} \Rightarrow_I 2 \quad \frac{\frac{\frac{\cancel{p}^2 \quad \cancel{q}^3}{p \wedge q} \wedge_I \quad (p \wedge q) \Rightarrow r^1}{r} \Rightarrow_I 3}{p \Rightarrow (q \Rightarrow r)} \Rightarrow_I 2}{[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]} \Rightarrow_E \equiv_I$$

(b) $(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]$

$$\frac{\frac{\frac{\cancel{r}^3 \quad r \Rightarrow p^2}{p} \Rightarrow_E \quad p \Rightarrow q^1}{q} \Rightarrow_E}{\frac{r \Rightarrow q}{r \Rightarrow q} \Rightarrow_I 3} \Rightarrow_I 2}{(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]} \Rightarrow_I 1$$

In two of the next three proofs we require a proof of the formula $(p \vee \neg p)$. This proof, for any formula p , has the form:

$$\frac{\frac{\frac{\neg p^2}{p \vee \neg p} \vee_I \quad \neg(p \vee \neg p)^1}{\perp} \perp_I \quad \frac{\frac{p^3}{p \vee \neg p} \vee_I \quad \neg(p \vee \neg p)^1}{\perp} \perp_I}{\frac{\perp}{p} \text{RAA } 2} \text{RAA } 3$$

$$\frac{\perp}{p \vee \neg p} \text{RAA } 1$$

It is elided as \vdots in the proofs below to save space.

(c) $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

$$\frac{\frac{\frac{\frac{\neg p^2 \quad p^3}{\perp} \perp_I}{\frac{\perp}{q} \perp_E} \perp_I \quad \frac{p \Rightarrow q \Rightarrow p^1}{p} \Rightarrow_I 3}{\frac{p \vee \neg p \quad p^2}{p} \vee_E 2} \Rightarrow_E \quad \frac{p}{[(p \Rightarrow q) \Rightarrow p] \Rightarrow p} \Rightarrow_I 1$$

This could also be done without the use of $p \vee \neg p$ as:

$$\frac{\frac{\frac{\frac{\neg p^2 \quad p^3}{\perp} \perp_I}{\frac{\perp}{q} \perp_E} \perp_I \quad \frac{p \Rightarrow q \Rightarrow p^1}{p} \Rightarrow_I 3}{\frac{p}{[(p \Rightarrow q) \Rightarrow p] \Rightarrow p} \Rightarrow_I 1} \neg_E \quad \frac{\perp}{[(p \Rightarrow q) \Rightarrow p] \Rightarrow p} \Rightarrow_I 1$$

(d) $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

$$\frac{\frac{\frac{\frac{p^2 \quad \neg r^3}{p \wedge \neg r} \wedge_I}{\perp} \perp_I \quad \frac{p \wedge \neg r^2}{\neg r} \wedge_E \quad \frac{\frac{p \wedge \neg r^2}{p} \wedge_E \quad p \Rightarrow r^1}{r} \perp_I}{\frac{r}{p \Rightarrow r} \Rightarrow_I 2} \text{RAA } 3}{\frac{\perp}{\neg(p \wedge \neg r)} \neg_I 2} \equiv_I 1$$

$$(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$$

(e) $(p \Rightarrow r) \equiv (\neg p \vee r)$

$$\frac{\frac{\frac{\frac{p^2 \quad \neg p^3}{\perp} \perp_I}{\frac{\perp}{r} \perp_E} \perp_I \quad \frac{p \vee \neg p \quad p^3}{p \vee \neg p} \vee_E 3}{\frac{r}{p \Rightarrow r} \Rightarrow_I 2} \vee_E 3}{\frac{\frac{\frac{\frac{p^2 \quad p \Rightarrow r^1}{r} \Rightarrow_E}{\neg p \vee r} \vee_I}{\neg p \vee r} \vee_I 2} \vee_E 2} \equiv_I 1$$

$$(p \Rightarrow r) \equiv (\neg p \vee r)$$