Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2001

Assignment #4 – Propositional Logic: Gentzen Sequent Calculus
Due 11:00am, Wednesday February 28, 2001

1. Give Classical Gentzen Sequent Calculus proofs of each of the following formulas (brackets are used in place of parentheses in some formulas to make the structure clearer):

   (a) $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

   (b) $(p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]$

   (c) $([(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$

   (d) $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

   (e) $(p \Rightarrow r) \equiv (\neg p \vee r)$

2. Consider the the intuitionistic sequent calculus in which the right hand side of the sequent is restricted to a single fomula (Greek letters denote sets of formulas, roman letters denote individual formulas):

$$\frac{}{\Gamma, A \longrightarrow A}\ id \qquad \frac{}{\Gamma, \bot \longrightarrow A}\ \bot$$

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge_L \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee_L \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee_{R_1} \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee_{R_2}$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C}\ \Rightarrow_L \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C}\ \neg_L \qquad \frac{\Gamma, A \longrightarrow \bot}{\Gamma \longrightarrow \neg A}\ \neg_R$$

   Prove that this version of the sequent calculus is sound (for the fragment of propositional calculus it covers) by showing that whenever there is a derivation of a sequent $\Gamma \longrightarrow A$ in this system, then there is a Natural Deduction proof of the formula A in which the open assumptions (non-discharged leaves) form a set $\Delta \subseteq \Gamma$.

   (Hint: Use complete induction on the height of the sequent calculus proof.)