Assignment #4 – Propositional Logic: Gentzen Sequent Calculus

Sample Solution

1. Give Gentzen Sequent Calculus proofs of each of the following formulas:

   (a) \[ ((p \land q) \Rightarrow r) \equiv [p \Rightarrow (q \Rightarrow r)] \]

   \[
   \frac{q, p \rightarrow r, p}{q, p \rightarrow r, p \land q} \quad \frac{q, p \rightarrow r}{q, p, r \rightarrow r} \quad \frac{q, p, r \rightarrow r}{\land_R} \quad \frac{q, p, (p \land q) \Rightarrow r \rightarrow r}{\Rightarrow_L} \quad \frac{p, (p \land q) \Rightarrow r \Rightarrow r}{\Rightarrow_R} \quad \frac{(p \land q) \Rightarrow r \rightarrow p \Rightarrow (q \Rightarrow r)}{\Rightarrow_R} \quad \frac{(p \land q) \Rightarrow r}{\equiv_R} \]

   \[
   \frac{p, q \rightarrow r, q}{p, q, r \rightarrow r} \quad \frac{p, q, r \rightarrow r}{p, q, p \Rightarrow (q \Rightarrow r) \rightarrow r} \quad \frac{p, q, p \Rightarrow (q \Rightarrow r) \rightarrow r}{\land_L} \quad \frac{p \Rightarrow (q \Rightarrow r) \rightarrow (p \land q) \Rightarrow r}{\Rightarrow_R} \]

   \[
   \frac{p, r \rightarrow (q \Rightarrow r)}{p \Rightarrow q \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)]} \equiv_R
   \]

   (b) \[ (p \Rightarrow q) \Rightarrow [(r \Rightarrow p) \Rightarrow (r \Rightarrow q)] \]

   \[
   \frac{r \rightarrow q, p, r}{r, r \Rightarrow p \rightarrow p \rightarrow q, p} \quad \frac{r, p \rightarrow q, p}{\Rightarrow_L} \quad \frac{r, r \Rightarrow p, p \Rightarrow q \rightarrow q}{\Rightarrow_R} \quad \frac{r \Rightarrow p, p \Rightarrow q \Rightarrow q \Rightarrow q}{\Rightarrow_R} \quad \frac{p \Rightarrow q \Rightarrow (r \Rightarrow p) \Rightarrow (r \Rightarrow q)}{\Rightarrow_R} \]

   (c) \[ ((p \Rightarrow q) \Rightarrow p) \Rightarrow p \]

   \[
   \frac{p \rightarrow q, p}{p \Rightarrow q, p} \quad \frac{p \Rightarrow q, p}{\Rightarrow_R} \quad \frac{p \Rightarrow p}{\Rightarrow_L} \quad \frac{p \Rightarrow (p \Rightarrow q) \Rightarrow p}{\Rightarrow_R} \]

   (d) \[ (p \Rightarrow r) \equiv \neg(p \land \neg r) \]

   \[
   \frac{p, \neg r \rightarrow p}{p, \neg r \rightarrow r} \quad \frac{p, r \rightarrow r}{\neg_R} \quad \frac{p, \neg r \rightarrow r}{\neg_L} \quad \frac{p, r \rightarrow \neg(p \land \neg r)}{\Rightarrow_R} \quad \frac{p, \neg r \rightarrow r \rightarrow \neg(p \land \neg r)}{\land_R} \quad \frac{p, \neg r \rightarrow r \rightarrow \neg(p \land \neg r)}{\Rightarrow_R} \quad \frac{p \Rightarrow r \rightarrow \neg(p \land \neg r)}{\equiv_R} \]

   \[
   \frac{p \Rightarrow r \rightarrow \neg(p \land \neg r)}{p \Rightarrow r \equiv \neg(p \land \neg r)}
   \]
(e) \((p \Rightarrow r) \equiv (\neg p \lor r)\)

\[
\begin{align*}
(p \Rightarrow r) & \Rightarrow (\neg p \lor r) \\
\frac{p \Rightarrow r, p}{p, p \Rightarrow r} & \Rightarrow L \\
\frac{p, p \Rightarrow r}{p \Rightarrow r} & \Rightarrow L \\
\frac{\neg p, p \Rightarrow r}{p \Rightarrow r} & \Rightarrow R \\
\frac{\neg p \lor r, p \Rightarrow r}{\neg p \lor r} & \Rightarrow R
\end{align*}
\]
2. Consider the following variation on the sequent calculus in which the right hand side of the sequent is restricted to a single formula (Greek letters denote sets of formulas, roman letters denote individual formulas):

\[
\begin{align*}
\Gamma, A \rightarrow A & \quad \text{id} \\
\Gamma, \bot \rightarrow A & \quad \bot \\
\Gamma, A, B \rightarrow C & \quad \land_L \\
\Gamma, A \land B \rightarrow C & \quad \land_R \\
\Gamma, A \rightarrow C & \quad \lor_L \\
\Gamma, A \lor B \rightarrow C & \quad \lor_R_1 \\
\Gamma, B \rightarrow C & \quad \lor_R_2 \\
\Gamma, A \rightarrow B \rightarrow C & \quad \Rightarrow_L \\
\Gamma, A \Rightarrow B \rightarrow C & \quad \Rightarrow_R \\
\Gamma, \neg A \rightarrow C & \quad \neg_L \\
\Gamma, A \rightarrow \bot \rightarrow C & \quad \neg_R \\
\end{align*}
\]

Prove that this version of the sequent calculus is sound (for the fragment of propositional calculus it covers) by showing that whenever there is a derivation of a sequent \( \Gamma \rightarrow A \) in this system, then there is a Natural Deduction proof of the formula \( A \) from open assumptions \( \Gamma \).

(Hint: Use complete induction on the height of the sequent calculus proof.)

**Proof:** The proof is by complete induction on the height of the intuitionistic Sequent Calculus proof. We assume that for all sequent calculus proofs of height less than \( n \) that the theorem holds. We now show that the theorem holds for any sequent calculus proof of height \( n \). The proof is by cases based on the final (i.e. bottom) rule of the Sequent Calculus proof.

- Suppose the last (and only) rule applied is of the form:

\[
\Gamma, A \rightarrow A \quad \text{id}
\]

Then we need to show that there is a Natural Deduction proof of \( A \) from open assumptions in the set \( \Gamma \cup \{A\} \). But:

\( A \)

is such a proof (in which \( A \) is both the conclusion and an open assumption).
• Suppose the last (and only) rule applied is of the form:

\[ \Gamma, \bot \rightarrow A^{id} \]

Then we need to show that there is a Natural Deduction proof of \( A \) from open assumptions in the set \( \Gamma \cup \{ \bot \} \). But:

\[ \vdash \Gamma \rightarrow \bot_E \]

is such a proof (in which \( A \) is the conclusion and \( \bot \) is the only open assumption).

• Suppose the proof is of the form:

\[ \Gamma, A, B \rightarrow C \]

Then we need to show that there is a Natural Deduction proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \land B \} \). But, by the induction hypothesis, since the proof of the upper sequent of the last rule is shorter than the overall proof (i.e. is of height less than \( n \)), there is a proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A, B \} \) of the form:

\[ \Gamma \vdash A \quad B \]

But then we may cap all leaves of that proof labeled with the propositions \( A \) and \( B \) with applications of the \( \land_E \) rule, as in:

\[ \Gamma \vdash A \land B \land_E \]

yielding a proof of the desired form.

(Note, that it is possible that \( A \), or \( B \), or both do not actually appear among the leaves of the Natural Deduction proof from the induction hypothesis. In that case, the construction simply omits the application of \( \land_E \) for that proposition, and the result still holds. This behavior will be assumed in the rest of the cases.)

• Suppose the proof is of the form:

\[ \Gamma \rightarrow A \quad \Gamma \rightarrow B \land_R \]

Then we need to show that there is a Natural Deduction proof of \( A \land B \) from open assumptions in the set \( \Gamma \). But, since the proofs of the upper sequents of the
bottom rule are both of height less than \( n \), by the induction hypothesis, there are proofs of \( A \) and \( B \) from open assumptions in the set \( \Gamma \) of the form:

\[
\begin{array}{c}
\vdots \\
A \quad \text{and} \quad B
\end{array}
\]

But then we may join those two proofs with an application of the \( \land_I \) rule, as in:

\[
\begin{array}{c}
\Gamma \quad \Gamma \\
\vdots \\
A \quad B
\end{array} \quad \begin{array}{c}
\land_I
\end{array}
\frac{A \land B}{\Gamma \quad \Gamma}
\]

yielding a proof of the desired form.

Note, it is not correct to say that the proofs of the upper sequents are of height \( n - 1 \). While one of them is of that height, the other may be of any height between 1 and \( n - 1 \) (since the proof tree is not necessarily balanced). Therefore, this proof requires strong induction, rather than weak induction.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\vdots \\
\Gamma, A \to C \\
\Gamma, B \to C
\end{array} \quad \begin{array}{c}
\lor_L
\end{array}
\frac{\Gamma, A \lor B \to C}{\Gamma, A \lor B \to C}
\]

Then we need to show that there is a Natural Deduction proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \lor B \} \). But, by the induction hypothesis, there are proofs of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \} \) and from open assumptions in the set \( \Gamma \cup \{ B \} \) of the form:

\[
\begin{array}{c}
\vdots \\
\Gamma \quad A \\
\vdots \\
\Gamma \quad B
\end{array}
\quad \begin{array}{c}
\lor_E
\end{array}
\frac{\Gamma \quad A \quad \Gamma \quad B}{\Gamma \quad A \lor B \quad \lor_E}
\]

But then we may join those two proofs with an application of the \( \lor_E \) rule, as in:

\[
\begin{array}{c}
\vdots \\
\Gamma \quad A \\
\vdots \\
\Gamma \quad B
\end{array}
\begin{array}{c}
\lor_E
\end{array}
\frac{A \lor B \quad \vdots \quad \vdots \quad C \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad C}{\Gamma \quad A \lor B \quad \lor_E}
\]

yielding a proof of the desired form.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\vdots \\
\Gamma \quad A_i
\end{array} \quad \begin{array}{c}
\lor_R
\end{array}
\frac{\Gamma \quad A_1 \lor A_2}{\Gamma \quad A_1 \lor A_2 \quad \lor_R}
\]
Then we need to show that there is a Natural Deduction proof of $A_1 \lor A_2$ from open assumptions in the set $\Gamma$. But, by the induction hypothesis, there is a proof of $A_i$ (for some $i \in \{1, 2\}$) from open assumptions in the set $\Gamma$ of the form:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
A_i
\end{array}
\]

But then we may terminate the proof with an application of the $\lor I$ rule, as in:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
A_i \\
A_1 \lor A_2
\end{array}
\]

yielding a proof of the desired form.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
\Gamma \rightarrow A \\
\Gamma, B \rightarrow C
\end{array}
\]

Then we need to show that there is a Natural Deduction proof of $C$ from open assumptions in the set $\Gamma \cup \{A \Rightarrow B\}$. But, by the induction hypothesis, there are proofs of $A$ from open assumptions in the set $\Gamma$ and of $C$ from open assumptions in the set $\Gamma \cup \{B\}$ of the form:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
A \\
B \\
\vdots \\
C
\end{array}
\]

But then we may cap all leaves of the proof of $C$ that are labeled with the proposition $B$ with an application of the $\Rightarrow E$ rule, as in:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
\Gamma, A \Rightarrow B \\
\vdots \\
\Gamma \rightarrow A \Rightarrow B
\end{array}
\]

yielding a proof of the desired form.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\Gamma, A \rightarrow B
\end{array}
\]

\[
\begin{array}{c}
\Gamma \rightarrow A \Rightarrow B
\end{array}
\]

\[
\begin{array}{c}
\rightarrow R
\end{array}
\]
Then we need to show that there is a Natural Deduction proof of $A \Rightarrow B$ from open assumptions in the set $\Gamma$. But, by the induction hypothesis, there is a proof of $B$ from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$
\begin{array}{c}
\Gamma \\
A \\
\vdots \\
B
\end{array}
$$

But then we may terminate the proof with an application of the $\Rightarrow_I$ rule, as in:

$$
\begin{array}{c}
\Gamma \\
A \\
\vdots \\
B \\
\hline
A \Rightarrow B
\end{array}
\Rightarrow_I
$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
A \quad \neg A \quad C
\end{array}
\neg_L
$$

Then we need to show that there is a Natural Deduction proof of $C$ from open assumptions in the set $\Gamma, \neg A$. But, by the induction hypothesis, there is a proof of $A$ from open assumptions in the set $\Gamma$ of the form:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
A
\end{array}
$$

But then we may terminate the proof with an application of the $\neg_E$ and $\bot_E$ rules, as in:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
A \\
\neg A \\
\hline
\bot 
\end{array}
\neg_E \bot_E
$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$
\begin{array}{c}
\Gamma, A \\
\bot \\
\hline
\neg A \\
\neg_R
\end{array}
$$

Then we need to show that there is a Natural Deduction proof of $\neg A$ from open assumptions in the set $\Gamma$. But, by the induction hypothesis, there is a proof of $\bot$ from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$
\begin{array}{c}
\Gamma \\
A \\
\vdots \\
\bot
\end{array}
$$
But then we may terminate the proof with an application of the $\neg\gamma$ rule, as in:

$$
\Gamma \quad A \\
\vdots \\
\downarrow \\
\neg A \quad \neg\gamma
$$

yielding a proof of the desired form.  

Q.E.D.