Relations

Recall the definition of a relation:

Definition: Given sets $S_1, \ldots, S_n$, an *n-ary relation* $\mathcal{R}$ is a subset of $S_1 \times \ldots \times S_n$.

A relation that is a subset of $A \times B$ will be called a *binary relation on (over, or between) A and B*. If the two carrier sets are the same, say A, the relation will be called a *binary relation on (or over) A*.

Instead of $(x, y) \in R$, we will sometimes write $R(x, y)$, or even xRy. The latter will be particularly used when the relation is denoted by a symbol, as in $x \leq y$.

Special Classes of Relations

Definition: A binary relation on a set A is *symmetric* iff:

For all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$

Definition: A binary relation on a set A is *reflexive* iff:

For all $x \in A$, $(x, x) \in R$

Definition: A binary relation on a set A is *transitive* iff:

For all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

Some Arbitrary Examples

Functions, Partial and Total

Definition[1]: The *domain* of a relation is the set

$$dom(R) = \{x \in A | \text{there is a } y \text{ such that } R(x, y)\}$$

Definition: The *range* of a relation is the set

$$range(R) = \{y \in B | \text{there is an } x \text{ such that } R(x, y)\}$$

Definition: A relation R between two sets A and B is *functional* iff each x in A is related to at most one y in B.

More Precisely:

[1] Most of the rest of the treatment of the Mathematical Preliminaries is based on Jean Gallier's text *Logic for Computer Science*, Harper & Row, 1986.

Functions, Partial and Total

Definition: A *partial function* is a triple

$$f = \langle A, G, B \rangle$$

where A and B are (possibly empty) sets and G is a (possibly empty) functional relation between A and B, called the *graph* of f, denoted $graph(f)$.

When no confusion can arise, i.e. in most circumstances, the function f and its graph are identified, and we will, therefore, talk about the domain and range of f.

A partial function $f = \langle A, G, B \rangle$ is often denoted as $f : A \to B$. For each element x in the domain of a partial function f, the unique element y in the range of f such that $(x, y) \in graph(f)$ is denoted by $f(x)$.

Definition: A partial function $f = \langle A, G, B \rangle$ is a *total function* iff the domain of f is all of A.

Equivalence Relations

Definition: A reflexive, symmetric, and transitive binary relation on a set A is called an *equivalence relation.*

Definition: Given an equivalence relation R on A, for each $x \in A$, the set $\{y \in A | (x, y) \in R\}$ is called the *equivalence class of x modulo R*, and is denoted $[x]_R$, or just $[x]$.

It should be apparent that for every $x' \in A$, if $x' \in [x]_R$, $[x']_R = [x]_R$.

Definition: The set of equivalence classes of A modulo R is called the *quotient of A by R*, denoted A/R.

The set A/R is a partition of A, since the classes are nonempty, any two are disjoint, and the union of the classes is A itself.

Definition: The surjective function $h_r : A \rightarrow A/r$ such that $h_r(x) = [x]_R$ is called the *canonical function* of R.

Orders, Partial and Total

Definition: A binary relation R on a set A is *antisymmetric* iff:

Definition: A reflexive, transitive, antisymmetric relation on a set A is called a *partial order* on A, often denoted by a symbol such as $\leq$.

Definition: If R is a partial order on A, and for every x and y in A either $R(x, y)$ or $R(y, x)$, then R is called a *total order*. This is also sometimes called a *linear order*.

Orders, Partial and Total

Definition: A transitive, antireflexive, antisymmetric relation on a set A is called a *strict order* on A, often denoted by a symbol such as $<$.

If I_A is the identity relation consisting of (x, x) for all x in A, then for any partial order $\leq$ over A, a corresponding strict order $<$ can be constructed as $\leq - I_A$.