Upper and Lower Bounds

Definition: Given a poset $(A, \leq)$, and given any subset X of A, an element $b \in A$ is a *lower bound* of X iff for all $x \in X, b \leq x$. An element $m \in A$ is an *upper bound* of X iff for all $x \in X, x \leq m$.

Note that b or m need not belong to X, and need not be unique. However, an upper or lower bound of X in X is unique.

Definition: A lower or upper bound of X in X is called *the least element* (or *the greatest element*) of X.

Such elements need not exist, but if they do, they are unique.

Minimal and Maximal Elements

Definition: Given a poset $(A, \leq)$, and given any subset X of A, an element $b \in X$ is a *minimal element of X* iff:

Definition: Given a poset $(A, \leq)$, and given any subset X of A, an element $b \in X$ is a *maximal element of X* iff:

Unlike least and greatest elements, minimal and maximal elements (if they exist) are not necessarily unique.

LUBs and GLBs

Definition: Given a poset $(A, \leq)$, and given any subset X of A, an element $m \in A$ is *the least upper bound of* X iff the set of upper bounds of X is nonempty, and m is the least element of this set.

Definition: Given a poset $(A, \leq)$, and given any subset X of A, an element $b \in A$ is *the greatest lower bound of* X iff the set of lower bounds of X is nonempty, and b is the greatest element of this set.

Can a set have an upper bound (resp. lower bound) and not have a least upper bound (resp. greatest lower bound)?

Sequences and Well-Founded Sets

Definition: Given two sets I and X, an *I-indexed sequence of elements of X* is any function $A : I \to X$, usually denoted by $(A_i)_{i \in I}$. The set I is called the *index set.*

Definition: A poset $(A, \leq)$, is *well-founded* iff it has no infinite decreasing sequence $(x_i)_{i \in \mathcal{N}}$. That is, if $<$ is the strict order associated with $\leq$, there is no sequence $(x_i)_{i \in \mathcal{N}}$ such that $x_{i+1} < x_i$ for all $i \geq 0$.

Well-Founded Sets

Lemma: Given a poset $(A, \leq)$, and a subset X of A, if X has no minimal element, then for every $x \in X$, there is a $y \in X$ such that $y < x$ (where $<$ is the strict order associated with $\leq$).

Proof:

Well-Founded Sets

Lemma: Given a poset $(A, \leq)$, $(A, \leq)$ is a well-founded set iff every non-empty subset of A has a minimal element.

Proof: