Existence of Answer Substitutions

We can ask the question whether there are larger families of formulas that will always produce answer substitutions. This turns out to be hard to answer in the context of resolution, and classical logic in general. Intuitionistic logic, however, yields some answers.

Recall the Intuitionistic formulation of Gentzen Sequent Calculus, which allows only one formula on the right of the sequent:

$$\frac{}{\Gamma, A \longrightarrow A}\ id \qquad \frac{}{\Gamma \longrightarrow \top}\ \top \qquad \frac{}{\Gamma, \bot \longrightarrow A}\ \bot \qquad \frac{\Gamma, A \longrightarrow}{\Gamma \longrightarrow \neg A}\ \neg_R$$

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge_L \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee_L \qquad \frac{\Gamma \longrightarrow A_i}{\Gamma \longrightarrow A_1 \vee A_2}\ \vee_R$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C}\ \Rightarrow_L \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B}\ \Rightarrow_R$$

The Existential Property

The key properties of intuitionistic reasoning are the *Disjunctive Property*:

> If $\vdash_{\mathcal{I}} \Phi \vee \Psi$ then either $\vdash_{\mathcal{I}} \Phi$ or $\vdash_{\mathcal{I}} \Psi$.

and the *Existential Property*:

> If $\vdash_{\mathcal{I}} \exists x(\Phi)$ then there is a term, t, such that $\vdash_{\mathcal{I}} \Phi[x := t]$

But these properties hold only for validity of the formula, not for logical consequence. We want something stronger.

The Existential Property and Horn Clauses

Notice that the earlier lemma tells us that Horn clause logic has an extended existential property:

> If Γ is a set of Horn Clauses, and $A_1, \ldots, A_n$ are atoms, then if $\Gamma \vdash \exists \bar{x}(A_1 \wedge \cdots \wedge A_n)$ then there is a term, t, such that $\Gamma \vdash A[x := t]$.

(Actually, this is *extended* in the sense that it covers consequences, but *restricted* in the sense that the subformula under the existential on the right is of a limited form.)

Note that this holds for both classical and intuitionistic provability. Horn clauses are so restricted in form that we can't really tell the difference.

Uniform Proofs and Horn Clauses

Can we get an analog to the completeness of linear input resolution for Horn clauses in the setting of sequent calculus proof search?

Definition: (Miller, et. al. 1987) We say that an intuitionistic sequent calculus proof is a *Uniform Proof* if, whenever the formula on the right hand side of the sequent is non-atomic, the sequent is the conclusion of an application of the right-hand proof rule for the principal operator of the formula.

If we think the formula on the right as the "goal" of the proof (since we want to prove it is a consequence of the formulas on the left), then looking at the construction of the proof from the bottom-up, we can say that a uniform proof is goal-directed, since you always decide what to do based on the form of the current goal.

Lemma: (Miller, 1987) If Γ is a set of Horn clauses, then the sequent $\Gamma \longrightarrow \exists\bar{x}(A_1 \wedge \cdots \wedge A_n)$ is provable if and only if it has a uniform proof.

The Existential Property - Harrop Formulas

Definition: The set of Harrop formulas is the smallest set such that:

- All atomic formulas are Harrop formulas
- If B is any formula, and D is a Harrop formula, then $B \Rightarrow D$ is a Harrop formula.
- If D is a Harrop formula, then $\forall x(D)$ is a Harrop formula.
- If D_1 and D_2 are Harrop formulas, then $D_1 \wedge D_2$ is a Harrop formulas.

Theorem (Harrop, 1960)**:** If Γ is a set of Harrop formulas, and B is any formula, then if $\Gamma \vdash_{\mathcal{I}} B$, the disjunctive and existential properties hold.

Corollary: If Γ is a set of Harrop formulas, and B is any formula, then the sequent $\Gamma \longrightarrow B$ is intuitionistically provable if and only if it has a proof that is uniform *in the last step*.

Hereditary Harrop Formulas

Definition: Consider the families of formulas defined by the following grammar:

$$
\begin{aligned}
D &:= A \mid \forall x(D) \mid D \wedge D \mid G \Rightarrow D \\
G &:= A \mid \forall x(G) \mid G \wedge G \mid D \Rightarrow G \mid \exists x(G) \mid G \vee G
\end{aligned}
$$

We call these the *hereditary Harrop formulas*, where D-formulas are refered to as (definite) clauses, and G-formulas are refered to as goals.

Corollary: If Γ is a set of D-formulas, and G is a G-formula, as above, then if $\Gamma \vdash_{\mathcal{I}} G$, the disjunctive and existential properties hold.

Theorem (Miller, 1987)**:** If Γ is a set of D-formulas, and G is a G-formula, then the sequent $\Gamma \longrightarrow G$ is intuitionistically provable if and only if it has a uniform proof.

Theorem (Harland, 1991)**:** The hereditary Harrop formulas are the largest family of formulas for which intuitionistic uniform proofs are complete.