Harvey Mudd College
Computer Science 80
Logic for Computer Science
Fall Semester 2002

Assignment #1 – Mathematical Preliminaries
Sample Solution

1. Using the directed graph representation used in class, provide examples of relations on a set of four objects that are:

   (There are many examples of each of these. Here is just one for each:)

   (a) transitive, not reflexive, not symmetric

   (b)transitive, symmetric, not reflexive

   (c) symmetric, reflexive, not transitive

   (d) anti-transitive, anti-symmetric, anti-reflexive

2. Some proofs concerning least elements:

   (a) If a subset X of a poset $(A, \leq)$ has a least element, is that element necessarily a greatest lower bound of X? If yes, prove it. If no, provide a counter example.

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   Yes, a least element is a greatest lower bound.

   Proof: (by contradiction)

   Suppose that $X \subseteq (A, \leq)$ has a least element l, but that l is not a greatest lower bound of X.

   A *least* element of a set is defined as a lower bound of the set residing within the set. So $l \in X$, and also in the (now obviously non-empty) set of lower bounds of X.

   Since l is not the greatest lower bound of X, there must, further, be at least one other lower bound b which is either greater than l or incomparable to it.

   But a lower bound of X must be less than or equal to (and therefore comparable to) every element of X, including l. Thus no such b can exist.

   (b) Prove that a totally ordered set with no least element is infinite in size.

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   First, a not very satisfying proof, which raises almost as many questions as it answers.

   Proof: (by contradiction)

   Suppose $(A, \leq)$ is a totally ordered set with no least element, but that it is finite.

   This means that the set has a number of distinct elements equal to some $n \in \mathcal{N}$. *(is that what finite means? we really never said that.)*

   If A has no least element, then since it is totally ordered this is the same as having no minimal element *(need to prove?)*. Therefore, by lemma proved in class, for every element there is a distinct smaller element.

   Given this fact we may pick an arbitrary element $a_0 \in A$ and make a series of n further selections to yield a chain of elements $a_0 \cdots a_n$ such that $a_{i+1} < a_i$.

   If these elements are all distinct, then by the pigeon-hole principle they cannot all reside in the set, contradicting our assumption that they had each been selected from A.

   If they are not distinct, then there are two elements a_i and a_j (with $i <_{\mathcal{N}} j$) which are equal. Then the elements $a_i \cdots a_j$ form a non-trivial cycle, which we prove in the next problem would contradict that $(A, \leq)$ is a partial order, let alone a total order.

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Next, a much better proof by induction. It shows that no natural number can be the size of a totally ordered set with no least element. This proof was inspired by one submitted by a student in solution to this problem.

Proof: (by induction)

Base Case: Assume the set $(S, \leq)$ is a totally ordered with no least element, and that the number of elements in S is 1.

Call the single element in S s. Clearly, $s \leq s$, and so s is a lower bound of S, and since it is in S it is by definition a least element of S, contradicting the assumption that S has no least element.

Thus a totally ordered set with no least element cannot be of size 1.

Induction Hypothesis: Assume that n cannot be the size of a totally ordered set with no least element.

Induction Step: Suppose $(S, \leq)$ is a totally ordered set of size $n + 1$ with no least element. Remove an arbitrary element s from S, yielding a set S' of size n. Clearly $(S', \leq)$ is also totally ordered. By the induction hypothesis, therefore, S' must have a least element; call it l.

Consider the relationship between l and s in the set S , of which both are members. Since S is totally ordered, either $l \leq s$ or $s \leq l$.

case 1 ($l \leq s$): Since l is also less than or equal to every other element of S (because it is a least element of S'), l is less than or equal to every element of S. Hence it is a lower bound of S, and, because it is in S, a least element of S.

case 2 ($s \leq l$): Pick an arbitrary element of S, call it t. If $t = s$, then $sleqt$. Otherwise, since $t \neq s$, t is an element of S'. Since l is a least element of S', $l \leq t$, and since $s \leq l$, by transitivity $s \leq t$. So, s is less than or equal to all elements of S, and is therefore a least element of S.

Therefore, in either case, we have identified a least element of S, contradicting its having no such element.

Thus, $n + 1$ cannot be the size of a totally ordered set with no least element.

3. Define a *non-trivial chain* in a binary relation R as a sequence $(a_1, \ldots, a_n)$ for some $n \geq 2$ such that the a_i are distinct, and $(a_i, a_{i+1}) \in R$ for $i \in [n-1]$. A chain is a *non-trivial cycle* if (a_n, a_1) is also an element of R.

   Prove that a relation is a partial order iff it is reflexive and transitive and has no non-trivial cycles.

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Proof: We prove each direction separately:

($\Leftarrow$) For this direction we must prove that if a relation is reflexive, transitive, and has no non-trivial cycles, then it is a partial order. We will do this by contradiction, assuming the existence of a relation that is reflexive, transitive, and has no non-trivial cycles, but which nevertheless is not a partial order.

Suppose R is reflexive and transitive and has no non-trivial cycles, but is not a partial order.

Since we are assuming it has two of the three features needed to be a partial order (reflexive and transitive), the only way R can fail to be a partial order is if it is not anti-symmetric. That is, that it has at least one symmetric pair or elements a, b such that $(a, b) \in R$, $(b, a) \in R$, and $a \neq b$.

Then, letting a_1 be a and a_2 be b, we see that (a_1, a_2) forms a (really small) non-trivial cycle, which contradicts our assumption that R had no non-trivial cycles.

($\Rightarrow$) In this direction we must prove that if a relation is a partial order, then it is reflexive, transitive, and has no non-trivial cycles. We will prove this by contradiction, supposing there is a partial order that does not have those three properties.

Suppose R is an arbitrary partial order which nevertheless is not all three of reflexive, transitive, and absent non-trivial cycles.

Since, by definition of being a partial order, R is reflexive and transitive, the only way the assumption can hold is that R has at least one non-trivial cycle. Pick one of those cycles, with elements named $(a_1, \ldots, a_n)$.

Claim: $(a_1, a_i) \in R$ for all $1 \leq i \leq n$.

By induction:

Base case ($i = 1$). $(a_1, a_i) = (a_1, a_1)$, which is in R since R is reflexive.

Induction step ($i \to i + 1$). Suppose $(a_1, a_j) \in R$ for all $1 \leq j \leq i$. Then since $(a_1, a_i) \in R$ by that assumption, and $(a_i, a_{i+1}) \in R$ as part of the non-trivial cycle, then, since R is transitive, $(a_1, a_{i+1}) \in R$.

Therefore, we know $(a_1, a_n) \in R$. Further, we know $(a_n, a_1) \in R$ by the definition of non-trivial cycle. Finally, $a_1 \neq a_n$ since the a_i are distinct. Together, these facts contradict the anti-symmetry of R. Therefore it is not possible for R to have non-trivial cycles.

4. Given a poset $(A, \leq)$, define the *lexicographic ordering*, $\ll$, on $A \times A$, induced by $\leq$, as follows:

   For all $x, y, x', y' \in A$, $(x, y) \ll (x', y')$ iff either:

   - $x = x'$ and $y = y'$, or
   - $x < x'$, or
   - $x = x'$ and $y < y'$

   Prove that if the poset $(A, \leq)$ is well-founded, then $(A \times A, \ll)$ is also well-founded. (You may assume [though you might want to confirm for yourself] that if $\leq$ is a partial order, then $\ll$ is indeed a partial order).

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Proof: There are a couple of ways to do this. We will use the lemma that a poset is well-founded iff every non-empty subset of it has a minimal element. It is also possible to work from the definition of well-founded poset.

Let $(A, \leq)$ be a well-founded poset. Then $(A \times A, \ll)$ is a poset (a fact which is annoying to prove, but not particularly difficult). If we can show that every non-empty subset of $A \times A$ has a minimal element, we will know that $(A \times A, \ll)$ is well-founded. With this in mind, let S be any non-empty subset of $A \times A$. We will try to find a minimal element of S.

Letting $X = \{x \mid (x, y) \in S\}$, we know X is non-empty, and $X \subseteq A$. Since A is well-founded, X must have a minimal element x_0. Similarly, letting $Y = \{y \mid (x_0, y) \in S\}$, we know Y is non-empty (why?), so Y has a minimal element y_0. Also, we can be sure $(x_0, y_0) \in S$ because of the way Y was defined. (If we had used $Y = \{y \mid (x, y) \in S\}$ instead, which might seem more obvious, this would not necessarily have been true.)

Claim: (x_0, y_0) is a minimal element of S.

Let $(a, b) \in S$, and suppose $(a, b) \ll (x_0, y_0)$. We need to show that $(a, b) = (x_0, y_0)$.

By the definition of $\ll$, either

- $a < x_0$, or
- $a = x_0$ and $b < y_0$, or
- $a = x_0$ and $b = y_0$

But $a \in X$ and x_0 is a minimal element of X, so it can't be that $a < x_0$.

If $a = x_0$, then $b \in Y$. But then since y_0 is a minimal element of Y, it can't be that $b < y_0$. So it's not true that $a = x_0$ and $b < y_0$.

Since the other two cases have been eliminated, it must be that $a = x_0$ and $b = y_0$, which is just what we needed to show.

Therefore, every non-empty subset S of $A \times A$ has a minimal element, so $(A \times A, \ll)$ is well-founded.

5. Ackerman's function on $\mathcal{N} \times \mathcal{N}$ is defined recursively in ML as:

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fun A(x,y) = if x = 0
               then y+1
               else if y = 0
                       then A(x-1,1)
                       else A(x-1, A(x,y-1));
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Assuming that such a recursive definition actually defines a partial function (this is non-trivial but is established in *recursive function theory*), prove by induction over the lexicographic ordering of $\mathcal{N} \times \mathcal{N}$ that Ackerman's function is in fact a total function on pairs of natural numbers.

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Proof: We will use the shorthand $A(x, y) \downarrow$ to indicate that Ackermann's function is defined for the pair (x, y). Also, let $\ll_s$ be the strict order associated with $\ll$ (which will be convenient later).

Since $(\mathcal{N} \times \mathcal{N}, \ll)$ is well-founded by the last problem, we can use the principle of complete induction show that $A(x, y) \downarrow$ for all $(x, y) \in \mathcal{N} \times \mathcal{N}$. Because of the structure of PCI, it is not necessary to state and prove a base case (though there is no harm in doing so, of course). The truth of the base cases is a consequence of the form of the statement of PCI and the fact that the set is well-founded.

The proof proceeds as follows:

Let $(x, y) \in \mathcal{N} \times \mathcal{N}$, and suppose that $A(a, b) \downarrow$ for all $(a, b) \ll_s (x, y)$. We need to show that $A(x, y) \downarrow$.

By cases:

($x = 0$) In this case, $A(x, y) = y + 1$, so certainly $A(x, y) \downarrow$.

($x > 0$ and $y = 0$) Since $(x - 1, 1) \ll_s (x, y)$, we know $A(x - 1, 1) \downarrow$. Since $A(x, y) = A(x - 1, 1)$, we know that $A(x, y) \downarrow$.

($x > 0$ and $y > 0$) Since $(x, y - 1) \ll_s (x, y)$, we know $A(x, y - 1) \downarrow$. No matter what $A(x, y - 1)$ is, we know $(x - 1, A(x, y - 1)) \ll_s (x, y)$, so $A(x - 1, A(x, y - 1)) \downarrow$. Since $A(x, y) = A(x - 1, A(x, y - 1))$, we know that $A(x, y) \downarrow$.

In each case, $A(x, y) \downarrow$, which was what we needed to show.

Therefore, by the principle of complete induction, $A(x, y) \downarrow$ for every $(x, y) \in \mathcal{N} \times \mathcal{N}$.