Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2002

Assignment #2 – Propositional Logic, Syntax and Semantics
Sample Solution

1. The proofs in this problem have as an immediate corollary that no proper prefix of a well-formed propositional formula is itself a well-formed propositional formula:

   Given a set of propositional letters, P, let Σ be the alphabet of the propositional language over P as defined in class. Define the function $K : \Sigma \rightarrow \mathcal{Z}$ as follows:

   - $K(\ (\ ) = -2$, $K(\ )\ ) = 2$
   - $K(\ p\ ) = 1$, for all $p \in P \cup \{\bot, \top\}$.
   - $K(\ \neg\ ) = 0$
   - $K(\ \wedge\ ) = K(\ \vee\ ) = K(\ \Rightarrow\ ) = K(\ \Leftarrow\ ) = K(\ \equiv\ ) = -1$

   Further define the function $K' : \Sigma^* \rightarrow \mathcal{Z}$ as $K'(u_1 \ldots u_n) = K(u_1) + \cdots + K(u_n)$.

   (a) Prove that for any $\Phi \in PROP$, $K'(\Phi) = 1$.

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We proceed by complete induction on the length of the formula.

Suppose $|\Phi| = n$ and, for all well-formed formulas Ψ such that $|\Psi| < n$, we know that $K'(\Psi) = 1$. There are two cases:

$|\Phi| = 1$: In that case, since Φ is a wff, then $\Phi = p$ for some $p \in P \cup \{\bot, \top\}$. But then, by definition, $K'(\Phi) = K'(p) = K(p) = 1$.

$|\Phi| > 1$: In that case, since Φ is a wff, then either $\Phi = (\neg\Psi)$ for some wff Ψ, or $\Phi = (\Psi_1 \diamond \Psi_2)$ for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$. We will deal we these two subcases individually.

$\Phi = (\neg\Psi)$ for some wff Ψ: In that case, clearly $|\Psi| < n$, and, by the induction hypothesis, $K'(\Psi) = 1$. Suppose that $\Psi = u_1 \cdots u_m$. Then $\Phi = (\neg u_1 \cdots u_m)$, and

$$\begin{aligned} K'(\Phi) &= K(() + K(\neg) + K(u_1) + \cdots + K(u_m) + K()) \\ &= K(() + K(\neg) + K'(\Psi) + K()) \\ &= -2 + 0 + 1 + 2 = 1 \end{aligned}$$

$\Phi = (\Psi_1 \diamond \Psi_2)$ for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$: In that case, clearly $|\Psi_1| < n$ and $|\Psi_2| < n$, and, by the induction hypothesis, $K'(\Psi_1) = K'(\Psi_2) = 1$. Further, suppose that $\Psi_1 = u_{11} \cdots u_{1m}$ and $\Psi_1 = u_{21} \cdots u_{2l}$. Then $\Phi = (u_{11} \cdots u_{1m} \diamond u_{21} \cdots u_{2l})$, and

$$\begin{aligned} K'(\Phi) &= K(() + K(u_{11}) + \cdots + K(u_{1m}) + K(\diamond) + \\ &\quad K(u_{21}) + \cdots + K(u_{2l}) + K()) \\ &= K(() + K'(\Psi_1) + K(\diamond) + K'(\Psi_2) + K()) \\ &= -2 + 1 + -1 + 1 + 2 = 1 \end{aligned}$$

So, in every case, we have shown that $K'(\Phi) = 1$. Q.E.D.

(b) Prove that for any $w \in \Sigma^*$ such that there is a $\Phi \in PROP$ such that w is a proper prefix of Φ, $K'(w) \neq 1$.

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We will, in fact, prove a stronger fact, that for any $w \in \Sigma^*$ such that there is a $\Phi \in PROP$ such that w is a proper prefix of Φ, $K'(w) < 1$. This is necessary to make the inductive argument go through.

While it may seem counter-intuitive, we will, again, proceed by induction on the length of Φ, ignoring the length of w.

Consider a particular arbitrary Φ and w and assume that the theorem holds for all proper prefixes of any wff Ψ such that $|\Psi| < |\Phi|$

We now examine by cases based on the structure of Φ. Since w must be a proper prefix of Φ, $|\Phi|$ must be greater than 1, so we need not consider the cases where Φ is a propositional letter or truth constant.

$\Phi = (\neg\Psi)$ for some wff Ψ: Then w can be of one of four forms:

$w = ($: Then $K'(w) = K(() = -2$

$w = (\neg$: Then $K'(w) = K(() + K(\neg) = -2 + 0 = -2$

$w = (\neg\Psi$: Then, assuming $\Psi = u_1 \cdots u_m$, we know by the previous problem that $K'(\Psi) = 1$ and we have that:

$$\begin{aligned} K'(w) &= K(() + K(\neg) + K(u_1) + \cdots + K(u_m) \\ &= K(() + K(\neg) + K'(\Psi) \\ &= -2 + 0 + 1 \\ &= -1 \end{aligned}$$

$w = (\neg w'$ for some w' a proper prefix of Ψ: Then, we know by the induction hypothesis (since $|\Psi| < |\Phi|$) that $K'(w') < 1$. Suppose that $w' = u_1 \cdots u_{m'}$ and that $K'(w') = x$. Then, we have that:

$$\begin{aligned} K'(w) &= K(() + K(\neg) + K(u_1) + \cdots + K(u'_m) \\ &= K(() + K(\neg) + K'(w') \\ &= -2 + 0 + x \\ &< -2 \qquad \text{(since } x < 1) \end{aligned}$$

$\Phi = (\Psi_1 \diamond \Psi_2)$ for some wffs Ψ_1, Ψ_2 and some $\diamond \in \{\wedge, \vee, \Rightarrow, \Leftarrow, \equiv\}$: Then w can be of one of six forms. (We will give a more abbreviated justification of each case than in the previous step.):

$w = ($: Then, as above, $K'(w) = K(() = -2$.

$w = (\Psi_1$: Then, $K'(w) = K(() + K'(\Psi) = -2 + 1 = -1$.

$w = (\Psi_1\diamond$: Then, $K'(w) = K(() + K'(\Psi_1) + K(\diamond) = -2 + 1 + -1 = -2$.

$w = (\Psi_1 \diamond \Psi_2$**:** Then, $K'(w) = K(() + K'(\Psi_1) + K(\diamond) + K'(\Psi_2) = -2 + 1 + -1 + 1 = -1$.

$w = (w'$ **for some** w' **a proper prefix of** Ψ_1**:** Then, as above, by the induction hypothesis we may assume that $K(w') = x < 1$, and, therefore, $K'(w) = K(() + K'(w') = -2 + x < -2$.

$w = (\Psi_1 \diamond w'$ **for some** w' **a proper prefix of** Ψ_2**:** Then, as above, by the induction hypothesis we may assume that $K(w') = x < 1$, and, therefore, $K'(w) = K(()+K'(\Psi_1)+K(\diamond)+K'(w') = -2+1+-1+x = -2+x < -2$.

So, in every case, we have shown that $K'(w) < 1$.

Q.E.D.

(c) Conclude that if $\Phi, \Psi \in PROP$, Φ then is not a proper prefix of Ψ.

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This is immediate: Since for every well-formed formula, Φ, $K'(\Phi) = 1$, and for every proper prefix Ψ of Φ, $K'(\Psi) < 1$, no proper prefix of Φ (which is an arbitrary WFF) can itself be a WFF.

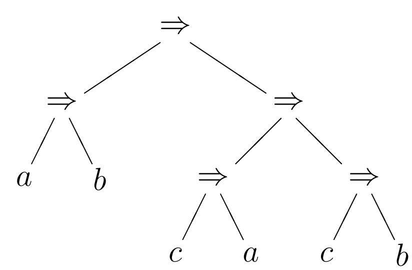
Q.E.D.

2. Expand the expression $\hat{v}((((a \wedge b) \Rightarrow c) \equiv (a \Rightarrow (b \Rightarrow c))))$ in terms of the function v and the various $h_\bullet$ functions.

$$\bullet\bullet\bullet$$

$$\begin{aligned}
\hat{v}((((a \wedge b) \Rightarrow c) \equiv (a \Rightarrow (b \Rightarrow c)))) &= h_\equiv(\hat{v}((a \wedge b) \Rightarrow c), \hat{v}(a \Rightarrow (b \Rightarrow c))) \\
&= h_\equiv(h_\Rightarrow(\hat{v}(a \wedge b), \hat{v}(c)), h_\Rightarrow(\hat{v}(a), \hat{v}(b \Rightarrow c))) \\
&= h_\equiv(h_\Rightarrow(h_\wedge(\hat{v}(a), \hat{v}(b)), \hat{v}(c)), h_\Rightarrow(\hat{v}(a), h_\Rightarrow(\hat{v}(b), \hat{v}(c)))) \\
&= h_\equiv(h_\Rightarrow(h_\wedge(v(a), v(b)), v(c)), h_\Rightarrow(v(a), h_\Rightarrow(v(b), v(c))))
\end{aligned}$$

3. Here are the tree and table for part b. The others are similar.



a	b	c	$a \Rightarrow b$	$c \Rightarrow a$	$c \Rightarrow b$	$(c \Rightarrow a) \Rightarrow (c \Rightarrow b)$	$(a \Rightarrow b) \Rightarrow ((c \Rightarrow a) \Rightarrow (c \Rightarrow b))$
True	True	True	True	True	True	True	True
True	True	False	True	True	True	True	True
True	False	True	False	True	False	False	True
True	False	False	False	True	True	True	True
False	True	True	True	False	True	True	True
False	True	False	True	True	True	True	True
False	False	True	True	False	False	True	True
False	False	False	True	True	True	True	True

4. Prove the following theorem from the lectures:

   Theorem (2.5.13)**:** Given formula $\Psi \in PROP$ and a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, if $U \models \Psi$ and there is an $1 \leq i \leq n$ such that Φ_i is valid, then $U - \{\Phi_i\} \models \Psi$.

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A slightly informal direct proof:

Since $U \models \Psi$, every valuation that satisfies all of U also satisfies Ψ. Since Φ_i is valid, it is satisfied by all valuations. Thus any valuation that satisfies U also satisfies $U - \Phi_i$, and vice versa. Thus any valuation which satisfies $U - \Phi_i$ satisifies U, and, in turn, Ψ. So $U - \Psi_i \models \Psi$.

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A somewhat more formal proof by contradiction:

Suppose not. Then there is a $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, a $\Psi \in PROP$, and a valid $\Phi_i \in U$ such that $U \models \Psi$, but $U - \Phi_i \not\models \Psi$. By the definition of of $\not\models$, there exists a valuation v such that $v' \models \Phi_j$ for all $\Phi_j \in U - \Phi_i$ but $v' \not\models \Psi$. Since Φ_i is valid, $v' \models \Phi_i$. But then $v' \models U$. So, $v' \models U$ but $v' \not\models \Psi$. This contradicts our assumption that $U \models \Psi$ (i.e that every model that satisfies U also satisfies Ψ).

5. Prove the following equivalences by appeal to the equivalences given in class (and in Figure 2.9 of the text). That is, give a series of rewritings of the formulas on the left and right until they arrive at the same point. (Or just go from left to right, if that seems more appropriate).

   (a) $(((a \wedge b) \Rightarrow c) \leftrightarrow (a \Rightarrow (b \Rightarrow c)))$

$$\begin{aligned}(a \wedge b) \Rightarrow c &\leftrightarrow \neg(a \wedge b) \vee c\\ &\leftrightarrow (\neg a \vee \neg b) \vee c\\ &\leftrightarrow \neg a \vee (\neg b \vee c)\\ &\leftrightarrow a \Rightarrow (\neg b \vee c)\\ &\leftrightarrow a \Rightarrow (b \Rightarrow c)\end{aligned}$$

   (b) $(((a \Rightarrow b) \Rightarrow a) \Rightarrow a) \leftrightarrow \top$

$$\begin{aligned}((a \Rightarrow b) \Rightarrow a) \Rightarrow a &\leftrightarrow ((\neg a \vee b) \Rightarrow a) \Rightarrow a\\ &\leftrightarrow (\neg(\neg a \vee b) \vee a) \Rightarrow a\\ &\leftrightarrow \neg(\neg(\neg a \vee b) \vee a) \vee a\\ &\leftrightarrow (\neg\neg(\neg a \vee b) \wedge \neg a) \vee a\\ &\leftrightarrow ((\neg a \vee b) \wedge \neg a) \vee a\\ &\leftrightarrow ((\neg a \vee b) \vee a) \wedge (\neg a \vee a)\\ &\leftrightarrow ((\neg a \vee b) \vee a) \wedge \top\\ &\leftrightarrow (\neg a \vee b) \vee a\\ &\leftrightarrow (b \vee \neg a) \vee a\\ &\leftrightarrow b \vee (\neg a \vee a)\\ &\leftrightarrow b \vee \top\\ &\leftrightarrow \top\end{aligned}$$

6. Prove that the set $\{\uparrow\}$ (NAND) is logically complete by giving formulas equivalent to each of the following that use only this one operator:

   There are many equivalent solutions.

   (a) $(\neg a)$

$$(a \uparrow a)$$

   (b) $(a \wedge b)$

$$(a \wedge b) \leftrightarrow \neg(\neg(a \wedge b)) \leftrightarrow \neg(a \uparrow b) \leftrightarrow ((a \uparrow b) \uparrow (a \uparrow b))$$

   (c) $(a \vee b)$

$$(a \vee b) \leftrightarrow \neg(\neg a \wedge \neg b) \leftrightarrow (\neg a \uparrow \neg b) \leftrightarrow ((a \uparrow a) \uparrow (b \uparrow b))$$

   (d) $(a \Rightarrow b)$

$$(a \Rightarrow b) \leftrightarrow \neg(a \wedge \neg b) \leftrightarrow (a \uparrow \neg b) \leftrightarrow (a \uparrow (b \uparrow b))$$

   (e) $(a \equiv b)$

$$\begin{aligned}(a \equiv b) \leftrightarrow ((a \Rightarrow b) \wedge (b \Rightarrow a)) &\leftrightarrow ((a \uparrow (b \uparrow b)) \wedge (b \uparrow (a \uparrow a)))\\ \leftrightarrow (((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a))) &\uparrow ((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a))))\end{aligned}$$