Satisfiability and Validity

Definitions: A propositional formula Φ is *satisfiable* if it has the value *true* in some interpretation. That is, if there is some valuation v such that $\hat{v}(\Phi) = true$. The satisfying interpretation is called a *model* of Φ, and we write $v \models \Phi$.

A formula $\Phi \in PROP$ is *valid*, written $\models \Phi$, if it is *true* in *all* interpretations. A valid formula is called a *tautology*.

Definitions: It is *falsifiable* (or *not valid*, or *invalid*) if it is *false* in some interpretation.

A formula $\Phi \in PROP$ is *unsatisfiable* (or *contradictory*) if it is not satisfiable. That is, if it is *false* in all interpretations (i.e. there is no valuation v such that $\hat{v}(\Phi) = true$).

Satisfiability and Validity

Theorem (2.5.3)**:** A formula $\Phi \in PROP$ is valid iff $(\neg\Phi)$ is unsatisfiable.

Proof. Almost by definition:

Mutual Satisfiability

The notion of satisfiability of a formula is extended to sets of formulas as follows:

Definition: A set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is *(mutually) satisfiable* if there is a valuation v such that $\hat{v}(\Phi_i) = \mathit{true}$, for all $1 \leq i \leq n$. The satisfying interpretation is called a *model* of U.

For example, the set $\{p, ((\neg p) \vee q), (q \vee r)\}$ is satisfiable:

Mutual Unsatisfiability

Definition: A set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is *(mutually) unsatisfiable* if for every valuation v there is some i $(1 \leq i \leq n)$ such that $\hat{v}(\Phi_i) = \mathit{false}$.

Note that for each valuation only one of the formulas in U need be *false*. Which formula is *false* can vary for each valuation.

For example, the set $\{p, ((\neg p) \vee q), (\neg p)\}$ is unsatisfiable:

Mutual Satisfiability

Theorem (2.5.6)**:** If a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is satisfiable, then for every $1 \leq i \leq n$ the set $U - \{\Phi_i\}$ is also satisfiable.

Proof.

Mutual Satisfiability

Theorem (2.5.7)**:** If a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is satisfiable, and the formula $\Psi \in PROP$ is valid, then the set $U \cup \{\Psi\}$ is also satisfiable.

Proof.

Mutual Unsatisfiability

Theorem (2.5.8)**:** If a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is unsatisfiable, then for any formula $\Psi \in PROP$ (valid or not), the set $U \cup \{\Psi\}$ is also unsatisfiable.

That is, if a set of formulas is unsatisfiable, there is nothing you can add to it to make it satisfiable. This is one side of the property of *monotonicity* of logic.

Proof.

Mutual Unsatisfiability

Theorem (2.5.9)**:** If a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\}$ is unsatisfiable, and some formula $\Phi_i \in U$ $(1 \leq i \leq n)$ is valid, then the set $U - \{\Phi_i\}$ is also unsatisfiable.

Proof.

Logical Consequence

Definition: Given a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, and a formula $\Psi \in PROP$, if Ψ is *true* in all models of U, then Ψ is a *logical consequence* of U, written $U \models \Psi$.

It is important to note that neither Ψ, nor the formulas in U, need be true in every interpretation. It is only necessary that those interpretations satisfying U (as a set) also satisfy Ψ (as a formula).

For example, $\{p, (\neg q)\} \models ((p \vee r) \wedge ((\neg q) \vee (\neg r))$:

p	q	r	$(\neg q)$	$(\neg r)$	$(p \vee r)$	$((\neg q) \vee (\neg r))$	$((p \vee r) \wedge ((\neg q) \vee (\neg r)))$
true	*true*	*true*					
true	*true*	*false*					
true	*false*	*true*					
true	*false*	*false*					
false	*true*	*true*					
false	*true*	*false*					
false	*false*	*true*					
false	*false*	*false*					

Note that a valid formula is a logical consequence of the empty set of formulas.

Logical Consequence

Logical consequence is a *meta-level property*, whereas implication is a defined operator. Nevertheless, the two are closely intertwined, as shown by the following theorem.

Theorem (2.5.11)**:** Given a formula $\Psi \in PROP$ and a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, $U \models \Psi$ iff $\models \Phi_1 \land \cdots \land \Phi_n \Rightarrow \Psi$.

Logical Consequence

As with mutual satisfiability, some useful properties can be shown:

Theorem (2.5.12)**:** Given formula $\Psi, \Xi \in PROP$ and a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, if $U \models \Psi$ then $U \cup \{\Xi\} \models \Psi$.

Proof.

Logical Consequence

Theorem (2.5.13)**:** Given formula $\Psi \in PROP$ and a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, if $U \models \Psi$ and there is an $1 \leq i \leq n$ such that Φ_i is valid, then $U - \{\Phi_i\} \models \Psi$.

Proof.

Thus, valid formulas don't contribute much information (which is our intuition). In fact, we have the following corollary:

Corollary: There is no falsifiable formula $\Psi \in PROP$ such that, given a set of tautologies $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, $U \models \Psi$.