First-Order Natural Deduction

Definition: The first order Natural Deduction System is derived from the propositional system by the addition of four rules:

$$\frac{\Phi}{\forall x(\Phi)}\,\forall_I * \qquad \frac{\forall x(\Phi)}{\Phi[x := t]}\,\forall_E$$

$$\frac{\Phi[x := t]}{\exists x(\Phi)}\,\exists_I \qquad \frac{\exists x(\Phi) \quad \begin{array}{c}\not\Phi \\ \vdots \\ \Psi\end{array}}{\Psi}\,\exists_E * *$$

with the following provisos:

* - The variable x may not occur free in any assumptions upon which the proof of Φ depends (i.e. in no undischarged assumption).

** - The variable x is not free in Ψ, or in any open assumption of the proof of Ψ other than the one discharged by the rule.

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Some good proofs:

$$\frac{}{\forall x(\forall y(p(x,y))) \Rightarrow \forall y(\forall x(p(x,y)))}$$

$$\frac{}{\forall x(p(x) \land q(x)) \Rightarrow (\forall x(p(x)) \land \forall x(q(x)))}$$

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Some good proofs:

$$\overline{\forall x(p(x)) \Rightarrow \exists x(p(x))}$$

$$\overline{\exists x(p(x) \vee q(x)) \Rightarrow (\exists x(p(x)) \vee \exists x(q(x)))}$$

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Why are the provisos important? Consider the following unsound proofs:

$$\overline{(0 = 0) \Rightarrow \forall x(x = 0)}$$

$$\overline{\forall x(\neg\forall y(x = y)) \Rightarrow (\neg\forall y(y = y))}$$

Inference in First-Order Natural Deduction

- If a bug is in a heated jar, its dead.
- This jar is heated.
- This bug is in this jar.
- This bug is dead.