Proof of Preliminary to Herbrand's Theorems

Theorem (3.9.6)**:** A closed set of clauses Γ has a model if and only if it has a Herbrand Model.

Proof. Clearly if the set of formulas has a Herbrand Model then it has a model (since a Herbrand Model is a model), so that direction is trivial. We will focus on the other direction:

If a set of clauses has a model, it has a Herbrand Model.

First recall that in any Herbrand interpretation of Γ the domain is fixed (as the Herbrand Universe of the Γ), and the interpretations of the constant and function symbols are fixed (as themselves). The only thing open to choice is the interpretation of the predicate symbols. We specify those interpretations by stating the subset of the Herbrand base that is true (all other atoms in the Herbrand base are assumed to be false).

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Now, since Γ has a model, suppose that the $\mathcal{L}$-structure $\mathcal{I}$ is a model of Γ.

Define the Herbrand Interpretation $\mathcal{H}$ by setting the elements in the following subset of the Herbrand Base to true:

$$\{ \; p_i(t_1, \ldots, t_n) \mid p_{i\mathcal{I}}(t_{1\mathcal{I}}, \ldots, t_{n\mathcal{I}}) = \mathit{true} \; \}$$

That is, we set to true in our model exactly those ground atoms that the model $\mathcal{I}$ sets to true. (Since we are concerned only about ground atoms, the t_i are all ground, so above we are ignoring the detail of applying the interpretations first to an assignment.)

We must now show that this interpretation, $\mathcal{H}$, is a model of Γ. For this to be so, $\mathcal{H}$ must be a model of each clause $\forall \bar{x}.C_i \in \Gamma$. Recall that to model a clause with universally bound variables, the underlying formula must be true in the interpretation for any assignment to the variables. That is, for each assignment s of values in the domain (which is the herbrand universe) to the variables, $C_{i\mathcal{H}}(s) = \mathit{true}$.

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Now, since $\mathcal{I}$ is a model of Γ, it is a model of each clause in Γ.

Therefore, for every clause $\forall\bar{x}.C_i \in \Gamma$, and every assignment s' (of values in the domain of $\mathcal{I}$) to the variables, $C_{i\mathcal{I}}(s') = \mathit{true}$.

Therefore, further, for each C_i and each assignment s', there is a literal $D_{ij} \in C_i$ such that $D_{ij\mathcal{I}}(s') = \mathit{true}$.

(Note that the particular literal in the clause that is true may vary depending on the assignment. We simply know that for each assignment there is at least one literal in each clause that is true.)

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Now, depending on the nature of $\mathcal{I}$ and Γ, some of the assignments may include assignments to the variables of domain elements that do not arise from terms.

For example, if the language of Γ is the language of natural number arithmetic and the domain in $\mathcal{I}$ is the integers, then a model must work even when variables are assigned domain values in the negative integers, which will never arise as the interpretation of ground terms in the language.

Such assignments will be of no use to us. But, since $\mathcal{I}$ is a model over all assignments, it is also a model in cases where the assignment assigns only domain values that do arise from terms.

Proof of Preliminary to Herbrand's Theorems

Fix an arbitrary clause C_i and an arbitrary assignment s of values in the Herbrand Universe to the variables of C_i. We must show that $C_{i\mathcal{H}}(s) = \mathit{true}$.

Suppose the assignment s assigns to each x_i the term (from the Herbrand Universe) t'_i. Consider the fixed assignment s' over $\mathcal{I}$ such that x_i is assigned the value d_i from the domain of $\mathcal{I}$, where $d_i = t'_{i\mathcal{I}}$ (the interpretation of the term t'_i in $\mathcal{I}$).

Since $\mathcal{I}$ is a model of C_i for all assignments, it is a model under this assignment. But, then, there is a literal $D_{ij} \in C_i$ such that $D_{ij\mathcal{I}}(s') = \mathit{true}$.

But, then, by the definition of $\mathcal{H}$, and our assumptions about the relationship between s, and s', since $D_{ij\mathcal{I}}(s') = \mathit{true}$, it must be that $D_{ij\mathcal{H}}(s) = \mathit{true}$, and therefore $C_{i\mathcal{H}}(s) = \mathit{true}$.

Q.E.D.