CS 181b, Advanced Algorithms
Spring 2002
Homework 8a
Due Thursday, March 28

1. **[10 Points] Deriving Amazing Formulae with Discrete Calculus!**

   (a) Use Discrete Calculus to derive a closed-form formula for

   $$\sum_{k=0}^{n} k^3.$$

   Show your work in detail. (Remember that using Stirling's Triangle will help you here!)

   (b) Use Discrete Calculus to derive a closed-form formula for

   $$\sum_{k=0}^{n} k^4.$$

   Show your work in detail.

2. **[15 Points] Stirling Numbers of the Second Kind!** In class we used "Stirling Numbers of the Second Kind" to help us express regular powers as the sum of falling powers. We defined Stirling's Triangle and showed how it was constructed. Specifically, let $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the Stirling number in the k^{th} column of row n of Stirling's Triangle. In class, we said that the rule for building Stirling's Triangle works like this:

   (a) $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$. (That is, the element at the top of the triangle is 1.)

   (b) $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0$ for all $n \geq 1$. (That is, the left edge of the triangle is all 0's.)

   (c) $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1$ for all $n \geq 1$. (That is, the right edge of the triangle is all 1's.)

   (d) $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ for any $n \geq 1$.

   This is very similar to the binomial coefficients $\binom{n}{k}$ in Pascal's Triangle. The Binomial coefficients have some special significance: $\binom{n}{k}$ is the number of ways to choose k objects from n distinct objects where order does not matter. Do the Stirling Numbers of the Second Kind have some meaning as well (other than being useful in Discrete Calculus)? Yes! It turns out that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ counts the number of different ways to partition n distinct objects into k nonempty sets. For example consider partitioning the three objects $1, 2, 3$ into two sets, neither of which is empty. There are only three ways to do this:

- $\{1, 2\}$, $\{3\}$.
- $\{1, 3\}$, $\{2\}$.
- $\{2, 3\}$, $\{1\}$.

Notice that $\left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} = 3$ (just look at Stirling's Triangle!).

(a) Given that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ counts the number of ways to partition n distinct objects into k nonempty sets, give a combinatorial argument that explains why $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$. (Show that both sides count the same thing in two different ways.)

(b) Give a combinatorial argument to explain why $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1$.

(c) Give a combinatorial argument to explain why $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$.

3. **[15 Points] More Fun with Discrete Calculus!**

(a) Let c be an arbitrary positive integer. What is $\Delta(c^x)$?.

(b) Great! Now use Discrete Calculus to show that

$$\sum_{k=0}^{n} c^k = \frac{c^{n+1} - 1}{c - 1}.$$

(We've proved this using the "xS-ively cute" technique and also by induction. Now we have a third proof!) Show your work in detail.

(c) Next, use Summation by Parts to compute the indefinite summation

$$\sum x 3^x \delta x.$$

Show your work in detail.

(d) Finally, use this result to find a simple closed-form formula for the summation

$$\sum_{k=0}^{n} k 3^k.$$

Verify that your formula is correct by trying a few different values of n.