Logic

Why Study Logic?

- A basis for computer hardware
- A basis for computer programming
- A basis for program optimization
- A basis for specification
- A basis for verification and testing

In a certain sense

Computing *is* Logic

Is all Logic Computing?

No, but

a lot of it can be reduced to computing.

Flavors of Logic

- Proposition Logic
- Predicate Logic

Studied in CS60

- Temporal Logic
- Modal Logics
- Programming Logics

Some exposure in CS*80*

- Fuzzy Logic

Some exposure in CS152 (Neural Networks)

Proposition Logic

- Also known as Switching Logic
- Basic elements are
  - 0 (false)
  - 1 (true)
  - proposition variables (take values 0 or 1)
  - either
    - functions (functional view)
    - connectives (expression view)

Mostly we use

- the function view
- and occasionally
  - the expression view

Proposition Logic Domain

- {false, true} (for purists)
  or
  {0, 1} (more readable)
  or
  $\{\bot, \top\}$ (more symmetric)

Proposition Logic Functions

- and
- or
- not
- implies
- iff (if, and only if)
- others

and

form 1 table:

x	y	and(x, y)
0	0	0
0	1	0
1	0	0
1	1	1

arguments results

and

- form 2 table:

and(x, y)	0	1
0	0	0
1	0	1

results

and

- rex “table”:
  - and(0, 0) => 0;
  - and(0, 1) => 0;
  - and(1, 0) => 0;
  - and(1, 1) => 1;

and

- shorter rex rules (using sequential convention):
  - and(1, 1) => 1;
  - and(x, y) => 0;

or

- form 1 table:

x	y	or(x, y)
0	0	0
0	1	1
1	0	1
1	1	1

or

- form 2 table:

or(x, y)	0	1
0	0	1
1	1	1

or

- rex “table”
  - or(0, 0) => 0;
  - or(0, 1) => 1;
  - or(1, 0) => 1;
  - or(1, 1) => 1;

or

- shorter rex rules:
  - or(0, 0) => 0;
  - or(x, y) => 1;

not

- form 1 table = form 2 table:

x	not(x)
0	1
1	0

not

- rex rules:
  - not(0) => 1;
  - not(1) => 0;

implies

- form 1 table:

x	y	implies(x, y)
0	0	1
0	1	1
1	0	0
1	1	1

implies

- form 2 table:

implies(x, y)	0	1
0	1	1
1	0	1

implies

- rex rules:
  - implies(0, 0) => 1;
  - implies(0, 1) => 1;
  - implies(1, 0) => 0;
  - implies(1, 1) => 1;

implies

- shorter rex rules (sequential):
  - implies(1, 0) => 0;
  - implies(x, y) => 1;

Concise Summary

(sequential convention applies)

- and(1, 1) => 1;
- and(x, y) => 0;
- or(0, 0) => 0;
- or(x, y) => 1;
- not(0) => 1;
- not(1) => 0;
- implies(1, 0) => 0;
- implies(x, y) => 1;

Expression Forms

- Use for greater readability of certain equalities
- Similar to ordinary discourse

Expression Forms

- Function symbols
  - and: $\wedge$ · && *juxtaposition*
    Example: x y means x $\wedge$ y
  - or: $\vee$ + ||
  - not: $\neg$! ' superscript ‾(overbar)
- Example: These mean the same thing:
  - $(a \wedge b) \vee (c \wedge \neg d)$
  - ab + cd'
- Binding order: not, and, or

Expression Forms

- Function symbols:
  - implies: $\rightarrow$ $\supset$
  - iff: $\equiv$ = ==

What We'll Use

- To start, we'll use
  $\wedge$ $\vee$ $\neg$ $\rightarrow$ $\equiv$
- When we discuss circuits, we'll use
  · + ' =

Logical Equivalences
(These can be shown by substituting all combinations of 0, 1 for variables)

- $a \wedge b = b \wedge a$
- $a \vee b = b \vee a$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $(a \vee b) \vee c = a \vee (b \vee c)$
- $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$

More Logical Equivalences

- $(a \wedge 0) = 0$
- $(a \wedge 1) = a$
- $(a \vee 0) = a$
- $(a \vee 1) = 1$

More Logical Equivalences

- $\neg(a \wedge b) = (\neg b \vee \neg a)$
- $\neg(a \vee b) = (\neg b \wedge \neg a)$

$\Big\}$ DeMorgan's Laws

- $(a \vee \neg a\, b) \quad = a \vee b$
- $(a \wedge (\neg a \vee b)) = a \wedge b$

Logical Equivalences for Implies

- $(a \rightarrow b) = (\neg a \vee b)$
- $(a \rightarrow b) = \neg(a \wedge \neg b)$
- $(0 \rightarrow b) = 1$
- $(1 \rightarrow b) = b$
- $(a \rightarrow 0) = \neg a$
- $(a \rightarrow 1) = 1$

More Logical Equivalences for Implies

- $(a \rightarrow bc) = (a \rightarrow b) \wedge (a \rightarrow c)$
- $((a \rightarrow b) \wedge (b \rightarrow c)) \rightarrow (a \rightarrow c)$
- $(a \rightarrow b) = (\neg b \rightarrow \neg a)$

Checking Relations using the **Boole-Shannon Principle**

- Relations hold iff they hold for any **substitution** of 0 and 1 for the variables (**uniformly** throughout the expression)
- Therefore, a relation holds if, choosing *any* variable V, it holds for V = 0 and for V = 1.
- But substituting 0 or 1 for a variable often yields **simplifications** that make the relation obvious.

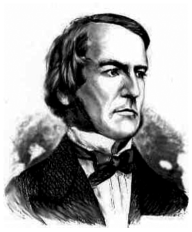
Example

- Verify $(a \rightarrow b) = (\neg b \rightarrow \neg a)$
- Choose a as the variable.
  - Substituting 0 for a:
    - $(0 \rightarrow b) = (\neg b \rightarrow \neg 0)$
  - which simplifies to:
    - $1 = (\neg b \rightarrow 1)$, a known equivalence
  - Substituting 1 for a:
    - $(1 \rightarrow b) = (\neg b \rightarrow \neg 1)$
  - which simplifies to:
    - $b = (\neg b \rightarrow 0)$

Boole and Shannon

- Boole
  - Invented "Boolean algebra" (switching theory)
  - (In modern mathematics, "Boolean algebra" is a more general, abstract, system)
- Shannon
  - Wrote thesis on switching theory
  - Invented "Information theory"
  - Maze-solving mouse
  - Wrote first chess-playing program
  - Wrote paper on the mathematics of juggling

Boole and Shannon



George Boole (1815-1864)

Claude Shannon (1916-2001)

Tautologies

- An expression that always evaluates to 1 (true) regardless of what value each variable is assigned is called a ***tautology***.
- The property of being a tautology can be checked using:
  - Truth-table construction
  - Boole-Shannon Principle, recursively
- Example of a tautology checker (applet): http://www.cs.hmc.edu/~keller/javaExamples/taut/taut.html