1. Using the directed graph representation used in class, provide examples of relations on a set of four objects that are:
   (There are many examples of each of these. Here is just one for each:)
   (a) transitive, not reflexive, not symmetric

   ![Graph Example](image1)

   (b) transitive, symmetric, not reflexive

   ![Graph Example](image2)

   (c) symmetric, reflexive, not transitive

   ![Graph Example](image3)

   (d) anti-transitive, anti-symmetric, anti-reflexive

   ![Graph Example](image4)
2. Define a non-trivial chain in a binary relation $R$ as a sequence $(a_1, \ldots, a_n)$ for some $n \geq 2$ such that the $a_i$ are distinct, and $(a_i, a_{i+1}) \in R$ for $i \in [n-1]$. A chain is a non-trivial cycle if $(a_n, a_1)$ is also an element of $R$.

Prove that a relation is a partial order if and only if it is reflexive and transitive and has no non-trivial cycles.

Proof: We prove each direction separately:

$(\Rightarrow)$ Suppose $R$ is a partial order. Then by definition $R$ is reflexive and transitive, so all we need to show is that $R$ has no non-trivial cycles. Suppose, to the contrary, that $(a_1, \ldots, a_n)$ is a non-trivial cycle in $R$.

Claim: $(a_1, a_i) \in R$ for all $1 \leq i \leq n$.

By induction:

Base case $(i = 1)$. $(a_1, a_1) = (a_1, a_1)$, which is in $R$ since $R$ is reflexive.

Induction step $(i \rightarrow i + 1)$. Suppose $(a_1, a_j) \in R$ for all $1 \leq j \leq i$.

Then since $(a_1, a_i) \in R$ by that assumption, and $(a_i, a_{i+1}) \in R$ as part of the non-trivial cycle, then, since $R$ is transitive, $(a_1, a_{i+1}) \in R$.

Therefore, we know $(a_1, a_n) \in R$. Further, we know $(a_n, a_1) \in R$ by the definition of non-trivial cycle. Finally, $a_1 \neq a_n$ since the $a_i$ are distinct. Together, these facts contradict the anti-symmetry of $R$. Therefore it is not possible for $R$ to have non-trivial cycles.

$(\Leftarrow)$ Suppose $R$ is reflexive and transitive and has no non-trivial cycles.

Claim: $R$ is anti-symmetric.

Suppose, to the contrary, that there exist $a, b \in R$ such that $(a, b) \in R$, $(b, a) \in R$, and $a \neq b$. Then, letting $a_1$ be $a$ and $a_2$ be $b$, we see that $(a_1, a_2)$ forms a (really small) non-trivial cycle. Therefore no such $a$ and $b$ exist, so $R$ is anti-symmetric.

Since we already know that $R$ is reflexive and transitive, it must be a partial order.
3. Given two binary relations, $R$ between $A$ and $B$, and $S$ between $B$ and $C$, their *composition*, denoted by $R \circ S$, is a relation between $A$ and $C$ defined by the set of ordered pairs $\{(a, c) | \text{there is a } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.

(a) Let $R = \{(a, b), (a, c), (c, d), (a, a), (b, a)\}$. What is the value of $R \circ R$?

\[ R \circ R = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c)\} \]

(There is a convenient way of composing binary relations by multiplying matrices of 0s and 1s. See if you can discover it.)

(b) Prove that, given any relation $R$ on a set $A$, $R$ is transitive iff $R \circ R$ is a subset of $R$.

\[ \text{Proof:} \text{ Again, we prove each direction separately:} \]

$(\Rightarrow)$ Suppose $R$ is transitive. Given any $(a, c) \in R \circ R$, we need to show $(a, c) \in R$. Now since $(a, c) \in R \circ R$, there must exist a $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R$. But then, since $R$ is transitive, $(a, c) \in R$.

$(\Leftarrow)$ Suppose $R \circ R \subseteq R$. Given $(a, b) \in R$ and $(b, c) \in R$, we need to show that $(a, c) \in R$. But by the definition of $R \circ R$, we must have $(a, c) \in R \circ R$. And, since $R \circ R \subseteq R$, $(a, c) \in R$ as well.
4. Given a poset \((A, \leq)\), define the lexicographic ordering, \(\ll\), on \(A \times A\), induced by \(\leq\), as follows:

For all \(x, y, x', y' \in A\), \((x, y) \ll (x', y')\) iff either:

- \(x = x'\) and \(y = y'\), or
- \(x < x'\), or
- \(x = x'\) and \(y < y'\)

Prove that if the poset \((A, \leq)\) is well-founded, then \((A \times A, \ll)\) is also well-founded.

(You may assume [though you might want to confirm for yourself] that if \(\leq\) is a partial order, then \(\ll\) is indeed a partial order).

\[\text{\vdots}\]

**Proof:** There are a couple of ways to do this. We will use the lemma that a poset is well-founded iff every non-empty subset of it has a minimal element. It is also possible to work from the definition of well-founded poset.

Let \((A, \leq)\) be a well-founded poset. Then \((A \times A, \ll)\) is a poset (a fact which is annoying to prove, but not particularly difficult). If we can show that every non-empty subset of \(A \times A\) has a minimal element, we will know that \((A \times A, \ll)\) is well-founded. With this in mind, let \(S\) be any non-empty subset of \(A \times A\). We will try to find a minimal element of \(S\).

Letting \(X = \{x \mid (x, y) \in S\}\), we know \(X\) is non-empty, and \(X \subseteq A\). Since \(A\) is well-founded, \(X\) must have a minimal element \(x_0\). Similarly, letting \(Y = \{y \mid (x_0, y) \in S\}\), we know \(Y\) is non-empty (why?), so \(Y\) has a minimal element \(y_0\). Also, we can be sure \((x_0, y_0) \in S\) because of the way \(Y\) was defined. (If we had used \(Y = \{y \mid (x, y) \in S\}\) instead, which might seem more obvious, this would not necessarily have been true.)

**Claim:** \((x_0, y_0)\) is a minimal element of \(S\).

Let \((a, b) \in S\), and suppose \((a, b) \ll (x_0, y_0)\). We need to show that \((a, b) = (x_0, y_0)\).

By the definition of \(\ll\), either

- \(a < x_0\), or
- \(a = x_0\) and \(b < y_0\), or
- \(a = x_0\) and \(b = y_0\)

But \(a \in X\) and \(x_0\) is a minimal element of \(X\), so it can’t be that \(a < x_0\).

If \(a = x_0\), then \(b \in Y\). But then since \(y_0\) is a minimal element of \(Y\), it can’t be that \(b < y_0\). So it’s not true that \(a = x_0\) and \(b < y_0\).

Since the other two cases have been eliminated, it must be that \(a = x_0\) and \(b = y_0\), which is just what we needed to show.

Therefore, every non-empty subset \(S\) of \(A \times A\) has a minimal element, so \((A \times A, \ll)\) is well-founded.
5. Ackerman’s function on $\mathbb{N} \times \mathbb{N}$ is defined recursively in ML as:

```ml
fun A(x,y) = if x = 0 then y+1
else if y = 0 then A(x-1,1)
else A(x-1, A(x,y-1));
```

Assuming that such a recursive definition actually defines a partial function (this is non-trivial but is established in recursive function theory), prove by induction over the lexicographic ordering of $\mathbb{N} \times \mathbb{N}$ that Ackerman’s function is in fact a total function on pairs of natural numbers.

\[\textbf{Proof:}\] We will use the shorthand $A(x,y) \downarrow$ to indicate that Ackermann’s function is defined for the pair $(x,y)$. Also, let $\ll_s$ be the strict order associated with $\ll$ (which will be convenient later).

Since $(\mathbb{N} \times \mathbb{N}, \ll)$ is well-founded by the last problem, we can use the principle of complete induction show that $A(x,y) \downarrow$ for all $(x,y) \in \mathbb{N} \times \mathbb{N}$. Because of the structure of PCI, it is not necessary to state and prove a base case (though there is no harm in doing so, of course). The truth of the base cases is a consequence of the form of the statement of PCI and the fact that the set is well-founded.

The proof proceeds as follows:

Let $(x,y) \in \mathbb{N} \times \mathbb{N}$, and suppose that $A(a,b) \downarrow$ for all $(a,b) \ll_s (x,y)$. We need to show that $A(x,y) \downarrow$.

By cases:

- $(x = 0)$ In this case, $A(x,y) = y + 1$, so certainly $A(x,y) \downarrow$.
- $(x > 0 \text{ and } y = 0)$ Since $(x-1,1) \ll_s (x,y)$, we know $A(x-1,1) \downarrow$. Since $A(x,y) = A(x-1,1)$, we know that $A(x,y) \downarrow$.
- $(x > 0 \text{ and } y > 0)$ Since $(x-1,y-1) \ll_s (x,y)$, we know $A(x-1,y-1) \downarrow$. No matter what $A(x,y-1)$ is, we know $(x-1,A(x,y-1)) \ll_s (x,y)$, so $A(x-1,A(x,y-1)) \downarrow$. Since $A(x,y) = A(x-1,A(x,y-1))$, we know that $A(x,y) \downarrow$.

In each case, $A(x,y) \downarrow$, which was what we needed to show.

Therefore, by the principle of complete induction, $A(x,y) \downarrow$ for every $(x,y) \in \mathbb{N} \times \mathbb{N}$. 