Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2002

Assignment #2 – Propositional Logic, Syntax and Semantics
Due 11:00am, Friday February 15, 2002

1. In computing the complexity of algorithms, it is often necessary to compute the potential size of a data set based on some parameter.

   (a) Assuming that the set P of proposisitional variables is a singleton set (that is there is only a single propositional variable available), how many well-formed formulas are there of lengths 1 through 10 (give the number of formulas for each length separately). Remember to count parentheses as part of the string length.

   (b) Given the same assumption as above, give a recursive definition of the formula for $t(n)$, the number of well-formed formulas of length n.

   (c) (extra credit) Provide a closed-form (that is, non-recursive) definition for the formula for $t(n)$.

2. The proofs in this problem have as an immediate corollary that no proper prefix of a well-formed propositional formula is itself a well-formed propositional formula:

   Given a set of propositional letters, P, let Σ be the alphabet of the propositional language over P as defined in class. Define the function $K : \Sigma \rightarrow \mathcal{Z}$ as follows:

   - $K(\,(\,) = -2$, $K(\,)\,) = 2$
   - $K(\,p\,) = 1$, for all $p \in P \cup \{\perp, \top\}$.
   - $K(\,\neg\,) = 0$
   - $K(\,\wedge\,) = K(\,\vee\,) = K(\,\Rightarrow\,) = K(\,\Leftarrow\,) = K(\,\equiv\,) = -1$

   Further define the function $K' : \Sigma^* \rightarrow \mathcal{Z}$ as $K'(u_1 \ldots u_n) = K(u_1) + \cdots + K(u_n)$.

   (a) Prove that for any $\Phi \in PROP$, $K'(\Phi) = 1$.
   (Hint: Proceed by complete induction on the length of the formula Φ.)

   (b) Prove that for any $w \in \Sigma^*$ such that there is a $\Phi \in PROP$ such that w is a proper prefix of Φ, $K'(w) \neq 1$.
   (Hint: Though it may seem counter intuitive, it is easiest to proceed again by complete induction on the length of Φ, ignoring the length of w, and to prove a slightly stronger property, that $K'(w) < 1$.)

   (c) Conclude that if $\Phi, \Psi \in PROP$, Φ then is not a proper prefix of Ψ.

3. For each of the following formulas, give the formation tree of the formula and a truth-table proof that the formula is a valid:

   (a) $((a \equiv b) \Rightarrow ((a \Rightarrow c) \Rightarrow (b \Rightarrow c)))$
   (b) $(((a \Rightarrow c) \wedge (b \Rightarrow c)) \Rightarrow ((a \vee b) \Rightarrow c))$
   (c) $(((a \Rightarrow b) \Rightarrow a) \Rightarrow a)$

   (Note: While I think you can probably make sense of why the first two formulas are valid by thinking about the meaning of the operators, that is probably not true of the third. That formula, which is called Pierce's Law, seems to contradict our notion of the meaning of implication. In fact, it is not a valid formula of intuitionistic logic, in which implication behaves much more according to our expectations.)

4. Prove the following theorem from the lectures:

   Theorem (2.5.12)**:** Given formulas $\Psi, \Xi \in PROP$ and an arbitrary set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, if $U \models \Psi$ then $U \cup \{\Xi\} \models \Psi$.

5. Prove the following equivalences by appeal to the equivalences given in class (and in Figure 2.9 of the text). That is, give a series of rewritings of the formulas on the left and right until they arrive at the same point. (Or just go from left to right, if that seems more appropriate).

   (a) $(((\neg((\neg a) \vee (\neg b))) \Rightarrow c) \leftrightarrow (a \Rightarrow ((\neg b) \vee c)))$
   (b) $(((a \Rightarrow b) \Rightarrow a) \Rightarrow a) \leftrightarrow \top$

6. Prove that the set $\{\uparrow\}$ (NAND) is logically complete by giving formulas equivalent to each of the following that use only this one operator:

   (a) $(\neg a)$
   (b) $(a \wedge b)$
   (c) $(a \vee b)$
   (d) $(a \Rightarrow b)$
   (e) $(a \equiv b)$