Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2002

Assignment #3 – Propositional (mostly) Logic: Natural Deduction
Due 5:00pm, Friday March 1, 2002

(A) Provide an encoding of the proof given in class that there exist two irrational numbers, a and b, such that a^b is rational, as a natural deduction proof. While we are focused on propositional logic, this uses a few features of first-order logic. You will need two additional rules from first-order natural deduction:

$$\frac{\forall x(\phi(x))}{\phi(t)}\ \forall_E \qquad \frac{\phi(x)}{\exists x(\phi(x))}\ \exists_I$$

The first rule says that if some property holds for some value of x, then it certainly holds for any given term t you substitute for x. For example, if we have the assumption $\forall x(\forall y((ra(x) \wedge (x = y)) \Rightarrow ra(y)))$ which says that if a number, y, is equal to a rational number, it is rational, we can use this to get the specific application of this assumption to two particular numbers used in the proof, as in:

$$\cfrac{\cfrac{\forall x(\forall y((ra(x) \wedge (x = y)) \Rightarrow ra(y)))}{\forall y((ra(2) \wedge (2 = y)) \Rightarrow ra(y))}\ \forall_E}{((ra(2) \wedge (2 = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})) \Rightarrow ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})}\ \forall_E$$

The second rule says that if you have a specific value that a property holds for, then you certainly know that there is some value the property holds for. This will be useful in reaching the desired conclusion in each of the case arguments. For example, one of the cases would terminate with:

$$\cfrac{\cfrac{\begin{matrix}\vdots \\ (ir(\sqrt{2}) \wedge ir(\sqrt{2})) \wedge ra(\sqrt{2}^{\sqrt{2}})\end{matrix}}{\exists y((ir(\sqrt{2}) \wedge ir(y)) \wedge ra(\sqrt{2}^{y}))}\ \exists_I}{\exists x(\exists y((ir(x) \wedge ir(y)) \wedge ra(x^{y})))}\ \exists_I$$

You may use any true equality or other obviously true property of arithmetic as an undischarged assumption in the proof. For example, that $\forall x(ir(x) \equiv (\neg ra(x)))$. Unlike in the natural language proof, it should be clear at the end exactly how what properties of arithmetic (like the one above about equality preserving rationality) are being used as assumptions.

(B) Give Natural Deduction proofs of each of the following formulas. That is, give a Natural Deduction proof tree with the given formula as its conclusion, and no undischarged assumptions. (Note, brackets are used in place of parentheses in some formulas to make the structure clearer).

Some of these proofs will require classical reasoning. In some cases cases it will be easiest using RAA. In others you may find that assuming $\phi \vee (\neg\phi)$ for some ϕ is easiest. If you choose to use the latter, then in the first place you do it you must expand the proof of that formula. For other uses you may just elide the proof by writing

$$\begin{array}{c}\vdots\\ \phi \vee (\neg\phi)\end{array}$$

1. $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$
2. $(p \vee q) \Rightarrow [((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r]$
3. $([(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$
4. $(p \Rightarrow r) \equiv (\neg p \vee r)$
5. $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

(C) In class we discussed that the "triple-xor" technique for exchanging the values of two program variables is sound if and only if the formula

$$((\Delta_{a_1} \wedge \ (\Delta_{b_1} \wedge \Delta_{a_2})) \Rightarrow ((a_2 \equiv b_o) \wedge (b_1 \equiv a)))$$

is valid (and provable), where Δ_{a_1}, Δ_{b_1}, and Δ_{a_2} are the formulas "defining" the values of the intermediate variables a_1, b_1, and a_2, given respectively as:

$$\begin{array}{lcl}(a_1 & \equiv & ((a_o \vee b_0) \wedge (\neg(a_0 \wedge b_0))))\\ (b_1 & \equiv & ((a_1 \vee b_0) \wedge (\neg(a_1 \wedge b_0))))\\ (a_2 & \equiv & ((a_1 \vee b_1) \wedge (\neg(a_1 \wedge b_1))))\end{array}$$

Here I have expanded out the definition of exclusive-or into primative connectives.

In constructing the proof of this formula in Natural Deduction, it is necessary to produce a sub-proof with the atomic formula b_1 as its conclusion and the formulas a_0, Δ_{a_1}, Δ_{b_1}, and Δ_{a_2} as open assumptions. Construct this sub-proof. As above, this proof requires classical reasoning, and you may use either RAA or an elided proof of a formula $\phi \vee \neg\phi$, as you prefer. (Indeed, I found one to be useful in one spot, and the other in another.)