

Harvey Mudd College
 Computer Science 80
 Logic for Computer Science
 Fall Semester 2001

Assignment #3 – Propositional (mostly) Logic: Natural Deduction
Sample Solution

(A) Complete the encoding of the proof given in class that there exist two irrational numbers, a and b , such that a^b is rational, as a natural deduction proof. While we are focused on propositional logic, this uses a few features of first-order logic. You will need two additional rules from first-order natural deduction:

$$\frac{\forall x(\phi(x))}{\phi(t)} \forall_E \qquad \frac{\phi(x)}{\exists x(\phi(x))} \exists_I$$

The first rule says that if some property holds for some value of x , then it certainly holds for any given term t you substitute for x . For example, if we have the assumption $\forall x(\forall y((ra(x) \wedge (x = y)) \Rightarrow ra(y)))$ which says that if a number, y , is equal to a rational number, it is rational, we can use this to get the specific application of this assumption to two particular numbers used in the proof, as in:

$$\frac{\frac{\forall x(\forall y((ra(x) \wedge (x = y)) \Rightarrow ra(y)))}{\forall y((ra(2) \wedge (2 = y)) \Rightarrow ra(y))} \forall_E}{((ra(2) \wedge (2 = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})) \Rightarrow ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}))} \forall_E$$

The second rule says that if you have a specific value that a property holds for, then you certainly know that there is some value the property holds for. This will be useful in reaching the desired conclusion in each of the case arguments. For example, one of the cases would terminate with:

$$\frac{\frac{\vdots}{(ir(\sqrt{2}) \wedge ir(\sqrt{2})) \wedge ra(\sqrt{2}^{\sqrt{2}})} \exists_I}{\exists y((ir(\sqrt{2}) \wedge ir(y)) \wedge ra(\sqrt{2}^y))} \exists_I}{\exists x(\exists y((ir(x) \wedge ir(y)) \wedge ra(x^y)))} \exists_I$$

You may use any true equality or other obviously true property of arithmetic as an undischarged assumption in the proof. Unlike in the natural language proof, it should be clear at the end exactly how what properties of arithmetic (like the one above about equality preserving rationality) are being used as assumptions.

In order to fit the proof on the page, two sub-proofs have been broken out and labeled as Ξ_1 and Ξ_2 .

$$\Xi_1 = \frac{-ra(\sqrt{2}^{\sqrt{2}}) \quad \frac{\forall x(\neg ra(x) \equiv ir(x))}{-ra(\sqrt{2}^{\sqrt{2}}) \equiv ir(\sqrt{2}^{\sqrt{2}})} \forall_E}{ir(\sqrt{2}^{\sqrt{2}})} \equiv_E$$

$$\Xi_2 = \frac{\frac{ra(2) \quad (2 = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})}{ra(2) \wedge (2 = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})} \wedge_I \quad \frac{\frac{\forall x(\forall y((ra(x) \wedge (x = y)) \Rightarrow ra(y)))}{\forall y((ra(2) \wedge (2 = y)) \Rightarrow ra(y))} \forall_E}{((ra(2) \wedge (2 = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})) \Rightarrow ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}))} \forall_E}{ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})} \wedge_I$$

$$\frac{\begin{array}{c} \vdots \\ ra(\sqrt{2}^{\sqrt{2}}) \vee \neg ra(\sqrt{2}^{\sqrt{2}}) \end{array}}{\Xi x(\Xi y((ir(x) \wedge ir(y)) \wedge ra(x^y)))} \vee_E^1 \quad \frac{\frac{\frac{\frac{\frac{ir(\sqrt{2}) \quad ir(\sqrt{2})}{ir(\sqrt{2}) \wedge ir(\sqrt{2})} \wedge_I \quad [ra(\sqrt{2}^{\sqrt{2}})]^1}{(ir(\sqrt{2}) \wedge ir(\sqrt{2})) \wedge ra(\sqrt{2}^{\sqrt{2}})} \wedge_I}{\Xi y((ir(\sqrt{2}) \wedge ir(y)) \wedge ra(\sqrt{2}^y))} \Xi_I}{\Xi x(\Xi y((ir(x) \wedge ir(y)) \wedge ra(x^y)))} \Xi_I \quad \frac{\frac{\frac{\frac{\frac{\Xi_1}{(ir(\sqrt{2}^{\sqrt{2}}) \quad ir(\sqrt{2}))}{(ir(\sqrt{2}^{\sqrt{2}}) \wedge ir(\sqrt{2}))} \wedge_I \quad \frac{\Xi_2}{ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})} \wedge_I}{(ir(\sqrt{2}^{\sqrt{2}}) \wedge ir(\sqrt{2})) \wedge ra((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})} \wedge_I}{\Xi y((ir(\sqrt{2}^{\sqrt{2}}) \wedge ir(y)) \wedge ra((\sqrt{2}^{\sqrt{2}})^y))} \Xi_I}{\Xi x(\Xi y((ir(x) \wedge ir(y)) \wedge ra(x^y)))} \Xi_I$$

(B) Give Natural Deduction proofs of each of the following formulas. That is, give a Natural Deduction proof tree with the given formula as its conclusion, and no undischarged assumptions. (Note, brackets are used in place of parentheses in some formulas to make the structure clearer).

Some of these proofs will require classical reasoning. In some cases it will be easiest using *RAA*. In others you may find that assuming $\phi \vee (\neg\phi)$ for some ϕ is easiest. If you choose to use the latter, then in the first place you do it you must expand the proof of that formula. For other uses you may just elide the proof by writing

$$\phi \dot{\vee} (\neg\phi)$$

1. $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

$$\frac{\frac{\frac{[q \wedge p]^2}{q} \wedge_E \quad \frac{\frac{[q \wedge p]^2}{p} \wedge_E \quad [p \Rightarrow (q \Rightarrow r)]^1}{q \Rightarrow r} \Rightarrow_E}{\frac{r}{(p \wedge q) \Rightarrow r} \Rightarrow_I 2} \Rightarrow_E \quad \frac{\frac{[p]^2 \quad [q]^3}{p \wedge q} \wedge_I \quad \frac{[(p \wedge q) \Rightarrow r]^1}{r} \Rightarrow_I 3}{\frac{q \Rightarrow r}{p \Rightarrow (q \Rightarrow r)} \Rightarrow_I 2} \Rightarrow_E}{\frac{r}{(p \wedge q) \Rightarrow r} \Rightarrow_I 2 \quad \frac{q \Rightarrow r}{p \Rightarrow (q \Rightarrow r)} \Rightarrow_I 2}{[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]} \equiv_I$$

2. $(p \vee q) \Rightarrow [((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r]$

$$\frac{\frac{[p \vee q]^1 \quad \frac{[p]^3 \quad \frac{[(p \Rightarrow r) \wedge (q \Rightarrow r)]^2}{p \Rightarrow r} \wedge_E}{r} \Rightarrow_E^3}{r} \wedge_E \quad \frac{[q]^3 \quad \frac{[(p \Rightarrow r) \wedge (q \Rightarrow r)]^2}{q \Rightarrow r} \wedge_E}{r} \wedge_E}{\frac{r}{((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r} \Rightarrow_I^2} \vee_E}{\frac{r}{((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r} \Rightarrow_I^2} \Rightarrow_I^1$$

In two of the next three proofs we require a proof of the formula $(p \vee \neg p)$. This proof, for any formula p , has the form:

$$\frac{\frac{\frac{[\neg p]^2}{p \vee \neg p} \vee_I \quad [\neg(p \vee \neg p)]^1}{\frac{\perp}{p} \text{ RAA } 2} \perp_I \quad \frac{\frac{[p]^3}{p \vee \neg p} \vee_I \quad [\neg(p \vee \neg p)]^1}{\frac{\perp}{\neg p} \neg_I 2} \text{ RAA } 3}{\frac{\perp}{p \vee \neg p} \text{ RAA } 1} \perp_I$$

It is elided as $\dot{\vee}$ in the proofs below to save space.

