Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2002

Assignment #4 – Propositional Logic: Gentzen Sequent Calculus
Sample Solution

1. Give Gentzen Sequent Calculus proofs of each of the following formulas:

   (a) $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q, p \longrightarrow r, p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q, p \longrightarrow r, q$}
\RightLabel{$\wedge_R$}
\BinaryInfC{$q, p \longrightarrow r, p \wedge q$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q, p, r \longrightarrow r$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$q, p, (p \wedge q) \Rightarrow r \longrightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$p, (p \wedge q) \Rightarrow r \longrightarrow q \Rightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$(p \wedge q) \Rightarrow r \longrightarrow p \Rightarrow (q \Rightarrow r)$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, q \longrightarrow r, p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, q \longrightarrow r, q$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, q, r \longrightarrow r$}
\BinaryInfC{$p, q, q \Rightarrow r \longrightarrow r$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$p, q, p \Rightarrow (q \Rightarrow r) \longrightarrow r$}
\RightLabel{$\wedge_L$}
\UnaryInfC{$p \wedge q, p \Rightarrow (q \Rightarrow r) \longrightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$p \Rightarrow (q \Rightarrow r) \longrightarrow (p \wedge q) \Rightarrow r$}
\RightLabel{$\equiv_R$}
\BinaryInfC{$\longrightarrow [(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$}
\end{prooftree}
$$

   (b) $(p \vee q) \Rightarrow [((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r]$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, q \Rightarrow r \longrightarrow p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, q \Rightarrow r, r \longrightarrow r$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$p, p \Rightarrow r, q \Rightarrow r \longrightarrow r$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q, p \Rightarrow r \longrightarrow q$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$q, p \Rightarrow r, r \longrightarrow r$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$q, p \Rightarrow r, q \Rightarrow r \longrightarrow r$}
\RightLabel{$\vee_L$}
\BinaryInfC{$p \vee q, p \Rightarrow r, q \Rightarrow r \longrightarrow r$}
\RightLabel{$\wedge_L$}
\UnaryInfC{$p \vee q, (p \Rightarrow r) \wedge (q \Rightarrow r) \longrightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$p \vee q \longrightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$\longrightarrow (p \vee q) \Rightarrow [((p \Rightarrow r) \wedge (q \Rightarrow r)) \Rightarrow r]$}
\end{prooftree}
$$

   (c) $([(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \longrightarrow q, p$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$\longrightarrow p \Rightarrow q, p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \longrightarrow p$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$(p \Rightarrow q) \Rightarrow p \longrightarrow p$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$\longrightarrow ((p \Rightarrow q) \Rightarrow p) \Rightarrow p$}
\end{prooftree}
$$

   (d) $(p \Rightarrow r) \equiv (\neg p \vee r)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \longrightarrow r, p$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, r \longrightarrow r$}
\RightLabel{$\Rightarrow_L$}
\BinaryInfC{$p, p \Rightarrow r \longrightarrow r$}
\RightLabel{$\neg_R$}
\UnaryInfC{$p \Rightarrow r \longrightarrow \neg p, r$}
\RightLabel{$\vee_R$}
\UnaryInfC{$p \Rightarrow r \longrightarrow \neg p \vee r$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p \longrightarrow r, p$}
\RightLabel{$\neg_L$}
\UnaryInfC{$p, \neg p \longrightarrow r$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$p, r \longrightarrow r$}
\RightLabel{$\vee_L$}
\BinaryInfC{$p, \neg p \vee r \longrightarrow r$}
\RightLabel{$\Rightarrow_R$}
\UnaryInfC{$\neg p \vee r \longrightarrow p \Rightarrow r$}
\RightLabel{$\equiv_R$}
\BinaryInfC{$\longrightarrow (p \Rightarrow r) \equiv (\neg p \vee r)$}
\end{prooftree}
$$

(e) $(p \Rightarrow r) \equiv \neg(p \wedge \neg r)$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{p, \neg r \longrightarrow p}\ id \qquad \dfrac{\dfrac{}{p, r \longrightarrow r}\ id}{p, \neg r, r \longrightarrow}\ \neg_L}{p, \neg r, p \Rightarrow r \longrightarrow}\ \Rightarrow_L
}{p \wedge \neg r, p \Rightarrow r \longrightarrow}\ \wedge_L
}{p \Rightarrow r \longrightarrow \neg(p \wedge \neg r)}\ \neg_R
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{p \longrightarrow r, p}\ id \qquad \dfrac{\dfrac{}{r, p \longrightarrow r}\ id}{p \longrightarrow r, \neg r}\ \neg_R}{p \longrightarrow r, p \wedge \neg r}\ \wedge_R
}{p, \neg(p \wedge \neg r) \longrightarrow r}\ \neg_L
}{\neg(p \wedge \neg r) \longrightarrow p \Rightarrow r}\ \Rightarrow_R
}{\longrightarrow (p \Rightarrow r) \equiv \neg(p \wedge \neg r)}\ \equiv_R
$$

2. Consider the following variation on the sequent calculus in which the right hand side of the sequent is restricted to a single fomula (Greek letters denote sets of formulas, roman letters denote individual formulas):

$$\frac{}{\Gamma, A \longrightarrow A}\ id \qquad \frac{}{\Gamma, \bot \longrightarrow A}\ \bot$$

$$\frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge_L \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee_L \qquad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee_{R_1} \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee_{R_2}$$

$$\frac{\Gamma \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \Rightarrow B \longrightarrow C}\ \Rightarrow_L \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \longrightarrow A}{\Gamma, \neg A \longrightarrow C}\ \neg_L \qquad \frac{\Gamma, A \longrightarrow \bot}{\Gamma \longrightarrow \neg A}\ \neg_R$$

Prove that this version of the sequent calculus is sound (for the fragment of propositional calculus it covers) by showing that whenever there is a derivation of a sequent $\Gamma \longrightarrow A$ in this system, then there is a Natural Deduction proof of the formula A from open assumptions Γ.

(Hint: Use complete induction on the height of the sequent calculus proof.)

Proof: The proof is by complete induction on the height of the intuitionistic Sequent Calculus proof. We assume that for all sequent calculus proofs of height less than n that the theorem holds. We now show that the theorem holds for any sequent calculus proof of height n. The proof is by cases based on the final (i.e. bottom) rule of the Sequent Calculus proof.

- Suppose the last (and only) rule applied is of the form:

  $$\frac{}{\Gamma, A \longrightarrow A}\ id$$

  Then we need to show that there is a Natural Deduction proof of A from open assumptions in the set $\Gamma \cup \{A\}$. But:

  $$A$$

  is such a proof (in which A is both the conclusion and an open assumption).

- Suppose the last (and only) rule applied is of the form:

$$\frac{}{\Gamma, \perp \longrightarrow A}\ id$$

Then we need to show that there is a Natural Deduction proof of A from open assumptions in the set $\Gamma \cup \{\perp\}$. But:

$$\frac{\perp}{A}\ \perp_E$$

is such a proof (in which A is the conclusion and $\perp$ is the only open assumption).

- Suppose the proof is of the form:

$$\frac{\begin{array}{c}\vdots \\ \Gamma, A, B \longrightarrow C\end{array}}{\Gamma, A \wedge B \longrightarrow C}\ \wedge_L$$

Then we need to show that there is a Natural Deduction proof of C from open assumptions in the set $\Gamma \cup \{A \wedge B\}$. But, by the induction hypothesis, since the proof of the upper sequent of the last rule is shorter than the overall proof (i.e. is of height less than n), there is a proof of C from open assumptions in the set $\Gamma \cup \{A, B\}$ of the form:

$$\begin{array}{c}\Gamma \quad A \quad B \\ \vdots \\ C\end{array}$$

But then we may cap all leaves of that proof labeled with the propositions A and B with applications of the $\wedge_E$ rule, as in:

$$\begin{array}{c}\Gamma \quad \dfrac{A \wedge B}{A}\ \wedge_E \quad \dfrac{A \wedge B}{B}\ \wedge_E \\ \vdots \\ C\end{array}$$

yielding a proof of the desired form.

(Note, that it is possible that A, or B, or both do not actually appear among the leaves of the Natural Deduction proof from the induction hypothesis. In that case, the construction simply omits the application of $\wedge_E$ for that proposition, and the result still holds. This behavior will be assumed in the rest of the cases.)

- Suppose the proof is of the form:

$$\frac{\begin{array}{c}\vdots \\ \Gamma \longrightarrow A\end{array} \quad \begin{array}{c}\vdots \\ \Gamma \longrightarrow B\end{array}}{\Gamma \longrightarrow A \wedge B}\ \wedge_R$$

Then we need to show that there is a Natural Deduction proof of $A \wedge B$ from open assumptions in the set Γ. But, since the proofs of the upper sequents of the bottom rule are both of height less than n, by the induction hypothesis, there are proofs of A and B from open assumptions in the set Γ of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma \\ \vdots \\ B \end{array}$$

But then we may join those two proofs with an application of the $\wedge_I$ rule, as in:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \begin{array}{c} \Gamma \\ \vdots \\ B \end{array}}{A \wedge B} \wedge_I$$

yielding a proof of the desired form.

Note, it is not correct to say that the proofs of the upper sequents are of height $n-1$. While one of them is of that height, the other may be of any height between 1 and $n-1$ (since the proof tree is not necessarily balanced). Therefore, this proof requires strong induction, rather than weak induction.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma, A \longrightarrow C \end{array} \quad \begin{array}{c} \vdots \\ \Gamma, B \longrightarrow C \end{array}}{\Gamma, A \vee B \longrightarrow C} \vee_L$$

Then we need to show that there is a Natural Deduction proof of C from open assumptions in the set $\Gamma \cup \{A \vee B\}$. But, by the induction hypothesis, there are proofs of C from open assumptions in the set $\Gamma \cup \{A\}$ and from open assumptions in the set $\Gamma \cup \{B\}$ of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ C \end{array} \quad \text{and} \quad \begin{array}{c} \Gamma \quad B \\ \vdots \\ C \end{array}$$

But then we may join those two proofs with an application of the $\vee_E$ rule, as in:

$$\dfrac{A \vee B \quad \begin{array}{c} \Gamma \quad \cancel{A} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \Gamma \quad \cancel{B} \\ \vdots \\ C \end{array}}{C} \vee_E$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma \longrightarrow A_i \end{array}}{\Gamma \longrightarrow A_1 \vee A_2} \vee_{R_2}$$

Then we need to show that there is a Natural Deduction proof of $A_1 \vee A_2$ from open assumptions in the set Γ. But, by the induction hypothesis, there is a proof of A_i (for some $i \in \{1, 2\}$) from open assumptions in the set Γ of the form:

$$\begin{array}{c}\Gamma \\ \vdots \\ A_i\end{array}$$

But then we may terminate the proof with an application of the $\vee_I$ rule, as in:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ A_i\end{array}}{A_1 \vee A_2}\ \vee_I$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \longrightarrow A\end{array} \quad \begin{array}{c}\vdots \\ \Gamma, B \longrightarrow C\end{array}}{\Gamma, A \Rightarrow B \longrightarrow C}\ \Rightarrow_L$$

Then we need to show that there is a Natural Deduction proof of C from open assumptions in the set $\Gamma \cup \{A \Rightarrow B\}$. But, by the induction hypothesis, there are proofs of A from open assumptions in the set Γ and of C from open assumptions in the set $\Gamma \cup \{B\}$ of the form:

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array} \qquad \text{and} \qquad \begin{array}{c}\Gamma \quad B \\ \vdots \\ C\end{array}$$

But then we may cap all leaves of the proof of C that are labeled with the proposition B with an application of the $\Rightarrow_E$ rule, as in:

$$\begin{array}{c}\Gamma \quad \dfrac{\begin{array}{c}\Gamma \\ \vdots \\ A\end{array} \quad A \Rightarrow B}{B}\ \Rightarrow_E \\ \vdots \\ C\end{array}$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma, A \longrightarrow B\end{array}}{\Gamma \longrightarrow A \Rightarrow B}\ \Rightarrow_R$$

Then we need to show that there is a Natural Deduction proof of $A \Rightarrow B$ from open assumptions in the set Γ. But, by the induction hypothesis, there is a proof of B from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ B \end{array}$$

But then we may terminate the proof with an application of the $\Rightarrow_I$ rule, as in:

$$\dfrac{\begin{array}{c} \Gamma \quad \cancel{A} \\ \vdots \\ B \end{array}}{A \Rightarrow B} \Rightarrow_I$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma \longrightarrow A \end{array}}{\Gamma, \neg A \longrightarrow C} \neg_L$$

Then we need to show that there is a Natural Deduction proof of C from open assumptions in the set $\Gamma, \neg A$. But, by the induction hypothesis, there is a proof of A from open assumptions in the set Γ of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

But then we may terminate the proof with an application of the $\neg_E$ and $\bot_E$ rules, as in:

$$\dfrac{\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \neg A}{\bot} \neg_E}{C} \bot_E$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma, A \longrightarrow \bot \end{array}}{\Gamma \longrightarrow \neg A} \neg_R$$

Then we need to show that there is a Natural Deduction proof of $\neg A$ from open assumptions in the set Γ. But, by the induction hypothesis, there is a proof of $\bot$ from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$\begin{array}{c} \Gamma \quad A \\ \vdots \\ \bot \end{array}$$

But then we may terminate the proof with an application of the $\neg_I$ rule, as in:

$$\dfrac{\begin{matrix} \Gamma \quad \cancel{A} \\ \vdots \\ \bot \end{matrix}}{\neg A}\ \neg_I$$

yielding a proof of the desired form.

Q.E.D.