Assignment #4 – Propositional Logic: Gentzen Sequent Calculus

Sample Solution

1. Give Gentzen Sequent Calculus proofs of each of the following formulas:

(a) \([p \land q \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]\)

\[
\begin{array}{c}
\frac{q, p \rightarrow r, p}{q, p \rightarrow r, p \land q} \text{id} \\
\frac{q, p \rightarrow r, q}{q, p, r \rightarrow r} \land_R \\
\frac{q, p, (p \land q) \Rightarrow r}{p, (p \land q) \Rightarrow r \rightarrow r} \Rightarrow_R \\
\frac{(p \land q) \Rightarrow r \rightarrow p \Rightarrow (q \Rightarrow r)}{\Rightarrow_R}
\end{array}
\]

\[
\Rightarrow_L \frac{q, p \rightarrow r, q}{p, q \Rightarrow r, q id} \\
\frac{p, q \Rightarrow r, q \Rightarrow r}{p, q \Rightarrow r id} \\
\frac{p, q \Rightarrow r}{p, q \Rightarrow r} \land_R \\
\frac{p \Rightarrow (q \Rightarrow r)}{p \Rightarrow (q \Rightarrow r) \rightarrow r \Rightarrow_R} \\
\frac{(p \land q) \Rightarrow r \rightarrow (p \land q) \Rightarrow r}{\Rightarrow_R}
\]

\[
\equiv_R \frac{(p \land q) \Rightarrow r}{\Rightarrow_L}
\]

(b) \((p \lor q) \Rightarrow \left[\left((p \Rightarrow r) \land (q \Rightarrow r)\right) \Rightarrow r\right]\)

\[
\begin{array}{c}
\frac{p, q \Rightarrow r \rightarrow p id}{p, p \Rightarrow r, q \Rightarrow r \rightarrow r} \rightarrow_R \\
\frac{q, p \Rightarrow r, q \Rightarrow r}{q, p, r \rightarrow r \rightarrow r} \Rightarrow_L \\
\frac{p \Rightarrow r, q \Rightarrow r}{p \lor q, (p \Rightarrow r) \land (q \Rightarrow r) \rightarrow r} \land_L \\
\frac{p \lor q \rightarrow \left((p \Rightarrow r) \land (q \Rightarrow r)\right) \Rightarrow r}{\Rightarrow_R}
\end{array}
\]

\[
\equiv_R \frac{(p \lor q) \Rightarrow \left[\left((p \Rightarrow r) \land (q \Rightarrow r)\right) \Rightarrow r\right]}{\Rightarrow_R}
\]

(c) \([(p \Rightarrow q) \Rightarrow p] \Rightarrow p\)

\[
\begin{array}{c}
\frac{q \Rightarrow p, p id}{p \Rightarrow q, p \Rightarrow R} \\
\frac{p \Rightarrow q, p \Rightarrow q}{p \Rightarrow R} \Rightarrow L \\
\frac{p \Rightarrow q \Rightarrow p \Rightarrow p}{p \Rightarrow R}
\end{array}
\]

\[
\Rightarrow_R \frac{(p \Rightarrow q) \Rightarrow p}{(p \Rightarrow q) \Rightarrow p \Rightarrow R}
\]

\[
\equiv_R \frac{(p \Rightarrow q) \Rightarrow p}{\Rightarrow_R}
\]

(d) \((p \Rightarrow r) \equiv (\neg p \lor r)\)

\[
\begin{array}{c}
\frac{p \rightarrow r, p id}{p, p \Rightarrow r \rightarrow r} \rightarrow_L \\
\frac{p, p \Rightarrow r \rightarrow r}{p \Rightarrow r \rightarrow \neg p, r \rightarrow r \Rightarrow_R} \\
\frac{p \Rightarrow r \rightarrow \neg p \lor r}{p \Rightarrow \neg p \lor r \lor_R}
\end{array}
\]

\[
\begin{array}{c}
\frac{p \rightarrow r, p \rightarrow r id}{p, \neg p \rightarrow r \rightarrow r \neg_L} \\
\frac{p, \neg p \rightarrow r \rightarrow r \neg_L}{p \rightarrow \neg p \lor r \lor_R}
\end{array}
\]

\[
\equiv_R \frac{(p \Rightarrow r) \equiv (\neg p \lor r)}{p \Rightarrow r \Rightarrow R}
\]
(e) \((p \Rightarrow r) \equiv \neg (p \land \neg r)\)
2. Consider the following variation on the sequent calculus in which the right hand side of the sequent is restricted to a single formula (Greek letters denote sets of formulas, roman letters denote individual formulas):

\[ \Gamma, A \rightarrow A \text{id} \quad \Gamma, \bot \rightarrow A \bot \]
\[ \Gamma, A, B \rightarrow C \quad \Gamma, A \rightarrow B \wedge_L \]
\[ \Gamma, A \rightarrow C \quad \Gamma, B \rightarrow C \quad \Gamma, A \rightarrow B \wedge_R \]
\[ \Gamma, A \rightarrow B \quad \Gamma \rightarrow A \vee B \quad \Gamma, A \rightarrow \bot \vee_{L_1} \quad \Gamma, A \rightarrow \bot \vee_{L_2} \]
\[ \Gamma, A \rightarrow \bot \quad \Gamma, \bot \rightarrow A \vee_R \]
\[ \Gamma, A \rightarrow \bot \quad \Gamma, A \rightarrow \bot \quad \Gamma, \neg A \rightarrow C \neg_L \quad \Gamma, A \rightarrow \bot \neg_R \]

Prove that this version of the sequent calculus is sound (for the fragment of propositional calculus it covers) by showing that whenever there is a derivation of a sequent \( \Gamma \rightarrow A \) in this system, then there is a Natural Deduction proof of the formula \( A \) from open assumptions \( \Gamma \).

(Hint: Use complete induction on the height of the sequent calculus proof.)

**Proof:** The proof is by complete induction on the height of the intuitionistic Sequent Calculus proof. We assume that for all sequent calculus proofs of height less than \( n \) that the theorem holds. We now show that the theorem holds for any sequent calculus proof of height \( n \). The proof is by cases based on the final (i.e. bottom) rule of the Sequent Calculus proof.

- Suppose the last (and only) rule applied is of the form:
  \[ \Gamma, A \rightarrow A \text{id} \]

  Then we need to show that there is a Natural Deduction proof of \( A \) from open assumptions in the set \( \Gamma \cup \{A\} \). But:
  \[ A \]

  is such a proof (in which \( A \) is both the conclusion and an open assumption).
• Suppose the last (and only) rule applied is of the form:

\[ \Gamma, \perp \rightarrow A \text{id} \]

Then we need to show that there is a Natural Deduction proof of \( A \) from open assumptions in the set \( \Gamma \cup \{ \perp \} \). But:

\[ \perp \rightarrow A \perp_E \]

is such a proof (in which \( A \) is the conclusion and \( \perp \) is the only open assumption).

• Suppose the proof is of the form:

\[ \vdots \]

\[ \Gamma, A, B \rightarrow C \]

\[ \vdots \]

\[ \Gamma, A \rightarrow C \rightarrow C \wedge L \]

Then we need to show that there is a Natural Deduction proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \wedge B \} \). But, by the induction hypothesis, since the proof of the upper sequent of the last rule is shorter than the overall proof (i.e. is of height less than \( n \)), there is a proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A, B \} \) of the form:

\[ \Gamma \rightarrow A \rightarrow B \rightarrow A \wedge B \rightarrow C \wedge L \rightarrow C \]

But then we may cap all leaves of that proof labeled with the propositions \( A \) and \( B \) with applications of the \( \wedge_E \) rule, as in:

\[ \vdots \]

\[ \Gamma \rightarrow A \wedge B \wedge E \rightarrow A \wedge B \wedge E \]

\[ \vdots \]

\[ \Gamma \rightarrow C \]

yielding a proof of the desired form.

(Note, that it is possible that \( A \), or \( B \), or both do not actually appear among the leaves of the Natural Deduction proof from the induction hypothesis. In that case, the construction simply omits the application of \( \wedge_E \) for that proposition, and the result still holds. This behavior will be assumed in the rest of the cases.)

• Suppose the proof is of the form:

\[ \vdots \]

\[ \Gamma \rightarrow A \rightarrow \Gamma \rightarrow B \rightarrow A \rightarrow B \rightarrow A \wedge B \rightarrow A \rightarrow B \rightarrow A \wedge B \rightarrow \wedge_R \]

Then we need to show that there is a Natural Deduction proof of \( A \wedge B \) from open assumptions in the set \( \Gamma \). But, since the proofs of the upper sequents of the
bottom rule are both of height less than \( n \), by the induction hypothesis, there are proofs of \( A \) and \( B \) from open assumptions in the set \( \Gamma \) of the form:

\[
\begin{align*}
\Gamma \quad \Gamma \\
\vdots & \quad \vdots \\
A & \quad \text{and} \quad B
\end{align*}
\]

But then we may join those two proofs with an application of the \( \land_I \) rule, as in:

\[
\begin{array}{c}
\Gamma \quad \Gamma \\
\vdots & \quad \vdots \\
A & \quad B \\
\hline
A \land B \quad \land_I
\end{array}
\]

yielding a proof of the desired form.

Note, it is not correct to say that the proofs of the upper sequents are of height \( n - 1 \). While one of them is of that height, the other may be of any height between 1 and \( n - 1 \) (since the proof tree is not necessarily balanced). Therefore, this proof requires strong induction, rather than weak induction.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\vdots \\
\Gamma, A \rightarrow C \\
\Gamma, B \rightarrow C \\
\hline
\Gamma, A \lor B \rightarrow C \quad \lor_L
\end{array}
\]

Then we need to show that there is a Natural Deduction proof of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \lor B \} \). But, by the induction hypothesis, there are proofs of \( C \) from open assumptions in the set \( \Gamma \cup \{ A \} \) and from open assumptions in the set \( \Gamma \cup \{ B \} \) of the form:

\[
\begin{align*}
\Gamma \quad A \\
\vdots \\
\hat{C} \\
\Gamma \quad B \\
\vdots \\
\hat{C}
\end{align*}
\]

But then we may join those two proofs with an application of the \( \lor_E \) rule, as in:

\[
\begin{array}{c}
\Gamma \quad A \\
\vdots \\
\hat{C} \\
A \lor B \\
\vdots \\
\hat{C} \\
\hline
C \quad \lor_E
\end{array}
\]

yielding a proof of the desired form.

• Suppose the proof is of the form:

\[
\begin{array}{c}
\vdots \\
\Gamma \rightarrow A_i \\
\Gamma \rightarrow A_1 \lor A_2 \quad \lor_{R_2}
\end{array}
\]
Then we need to show that there is a Natural Deduction proof of \( A_1 \lor A_2 \) from open assumptions in the set \( \Gamma \). But, by the induction hypothesis, there is a proof of \( A_i \) (for some \( i \in \{1, 2\} \)) from open assumptions in the set \( \Gamma \) of the form:

\[
\Gamma \\
\vdots \\
A_i
\]

But then we may terminate the proof with an application of the \( \lor I \) rule, as in:

\[
\Gamma \\
\vdots \\
A_i \quad \frac{A_1 \lor A_2}{A_1 \lor A_2} \quad \lor I
\]

yielding a proof of the desired form.

- Suppose the proof is of the form:

\[
\frac{\vdots}{\Gamma \Rightarrow A} \quad \frac{\vdots}{\Gamma, B \Rightarrow C} \quad \Rightarrow L
\]

Then we need to show that there is a Natural Deduction proof of \( C \) from open assumptions in the set \( \Gamma \cup \{A \Rightarrow B\} \). But, by the induction hypothesis, there are proofs of \( A \) from open assumptions in the set \( \Gamma \) and of \( C \) from open assumptions in the set \( \Gamma \cup \{B\} \) of the form:

\[
\Gamma \\
\vdots \\
A \quad \text{and} \\
\vdots \\
C
\]

But then we may cap all leaves of the proof of \( C \) that are labeled with the proposition \( B \) with an application of the \( \Rightarrow E \) rule, as in:

\[
\Gamma \\
\vdots \\
A \Rightarrow B \quad \Rightarrow E
\]

yielding a proof of the desired form.

- Suppose the proof is of the form:

\[
\frac{\vdots}{\Gamma, A \Rightarrow B} \quad \frac{\vdots}{\Gamma \Rightarrow A \Rightarrow B} \quad \Rightarrow R
\]
Then we need to show that there is a Natural Deduction proof of $A \Rightarrow B$ from open assumptions in the set $\Gamma$. But, by the induction hypothesis, there is a proof of $B$ from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$
\begin{array}{c}
\Gamma \ A \\
\vdots \\
B
\end{array}
$$

But then we may terminate the proof with an application of the $\Rightarrow I$ rule, as in:

$$
\begin{array}{c}
\Gamma \ A \\
\vdots \\
B
\hline
A \Rightarrow B \ \Rightarrow I
\end{array}
$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$
\begin{array}{c}
\vdots \\
\Gamma \to A \\
\Gamma, \neg A \to C \neg L
\end{array}
$$

Then we need to show that there is a Natural Deduction proof of $C$ from open assumptions in the set $\Gamma, \neg A$. But, by the induction hypothesis, there is a proof of $A$ from open assumptions in the set $\Gamma$ of the form:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
A
\end{array}
$$

But then we may terminate the proof with an application of the $\neg E$ and $\bot E$ rules, as in:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
A \neg A \neg E \\
\bot \neg E
\hline
C \bot E
\end{array}
$$

yielding a proof of the desired form.

- Suppose the proof is of the form:

$$
\begin{array}{c}
\vdots \\
\Gamma, A \to \bot \\
\Gamma \to \neg A \neg R
\end{array}
$$

Then we need to show that there is a Natural Deduction proof of $\neg A$ from open assumptions in the set $\Gamma$. But, by the induction hypothesis, there is a proof of $\bot$ from open assumptions in the set $\Gamma \cup \{A\}$ of the form:

$$
\begin{array}{c}
\Gamma \ A \\
\vdots \\
\bot
\end{array}
$$
But then we may terminate the proof with an application of the \( \neg I \) rule, as in:

\[
\begin{array}{c}
\Gamma \quad \mathcal{A} \\
\vdots \\
\downarrow \\
\neg \mathcal{A} \quad \neg I \\
\end{array}
\]

yielding a proof of the desired form. 

Q.E.D.