Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2001

Homework 6 – Predicate Logic: Semantics and Encoding
Sample Solution

1. Questions of Syntax:

   (a) Give the set of free variables and and the set of bound variables of the formula:

   $$\forall z(p(x) \land \forall x[\exists y\{q(y, f(z)) \Rightarrow r(g(g(w, x), f(y)))\}])$$

   • • •

   Free Variables: $\{x, w\}$
   Bound Variables: $\{x, y, z\}$

   (b) Give the result of substituting $f(y)$ for the variable x in that formula.

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   $$\forall z(p(f(y)) \land \forall x[\exists y\{q(y, f(z)) \Rightarrow r(g(g(w, x), f(y)))\}])$$

   The quantification over x in the right conjunct blocks substitution in that subformula.

   (c) Give the result of substituting $f(y)$ for the variable w in the original formula (not in the answer from the last substitution).

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   $$\forall z(p(x) \land \forall x[\exists v\{q(v, f(z)) \Rightarrow r(g(g(f(y), x), f(v)))\}])$$

   We must rename the bound uses of y so that we may substitute $f(y)$ under the binder without risk of capturing the y in that term.

2. Questions of Semantics:

For each of the following formulas (or, sets of formulas) show that they are both satisfiable and falsifiable, by giving interpretations for each case. Make sure you provide an assignment for quantified variables as well, where that appears necessary. Explanatory prose may also be helpful. For example:

- $\forall x(\exists y(l(y, x)))$

  Satisfiable: Let the domain be the integers, and interpret l as the less than relation. Clearly for every integer there is an integer less than it.
  (Alternately, let the domain be any non-empty set, and interpret l as any equivalence relation on that set. Since an equivalence relation is reflexive, then each element is at least related to itself, if nothing else.)

  Falsifiable: Let the domain be the natural numbers and interpret l as the less than relation. Then if x is 0, there is no smaller natural number.

Note: I begin with some "normal" interpretations. Then, I start picking some pretty arbitrary ones, just to remind you that ALL the interpretations would need to work for the formulas to be valid.

(a) $\exists x(\forall y(p(x, y)))$

Satisfiable: Let the domain be the natural numbers, and interpret p as the greater-than-or-equal relation. Then the assignment assigning zero to x satisfies the formula.

Falsifiable:As above, but interpret p as the greater-than relation. There is no number sunch that every number (including itself) is greater than it.

(b) $\forall x(p(x) \Rightarrow q(x))$

Satisfiable: Let the domain be 5 digit natural numbers. Interpret q such that it is true for numbers in the range 90000 to 99999 and false otherwise. Interpret p as true when the number is a valid zip code for a property in California and false otherwise. Then the formula is satisfied.

Falsifiable: Let the domain be the natural numbers, interpret p as true for odd numbers only, and q as true for prime numbers only. The formula is false, since 9 is odd but not prime.

(c) $(q(a) \wedge q(b) \wedge p(a)) \Rightarrow \forall x(p(x) \Rightarrow q(x))$

Satisfiable: Let the domain be the alphabetic characters. Interpret p as true only for the letters in the string "MICHAEL", q as true if a letter is before the letter 'P' in the alphabet, a as the letter 'H' and b as the letter 'G'. Then the formula is satisfied since all letters in that name are in the first half of the alphabet. (In fact, the interpretation of a and b are incosequential, since the universal is true regardless.)

Falsifiable: Let the domain be as above, and let p be true for letters in the string "MICHAEL ERLINGER". Let q be true if a letter is after 'P' in the alphabet, and let a be 'R' and b be 'S'. Then the conjunction is satisfied, but the universal is false, since 'A' is in the name, but is not after 'P' in the alphabet.

(d) $(heated(j) \wedge bug(b) \wedge in(b, j)) \Rightarrow dead(b)$

Satisfiable: Let the domain be things in my kitchen. Interepret the predicates as indicated by their names. Let j be the pot on the stove, and b the dead fly floating in the soup. Then the formula is true.

Falsifiable: Let the domain be things in my kitchen. Interepret *heated* as being below 32°F, *bug* as true for beef products, *in* as true iff the first thing is heavier than the second thing, and *dead* as true for things that I like to eat with peanut butter. Let j be a pie crust in my freezer, and b be a Hebrew National hot dog in the refrigerator. Then the formula is false since the pie crust is frozen, the hot dog is beef and lighter than the pie crust, yet I don't care for hot dogs with peanut butter.

(e) $$\begin{aligned}&(\forall x(plus(x, 0) \doteq x \wedge \\ &\forall x(\forall y(plus(s(x), y) \doteq s(plus(x, y))))) \Rightarrow \\ &\quad plus(s(s(0)), s(s(0))) \doteq s(s(s(s(0))))\end{aligned}$$

Satisfiable: Let the domain be the natural numbers. Interpret 0 as zero, s as the successor function, and *plus* as the addition function. (Note, the interpretation of $\doteq$ is always fixed to be equality on the underlying domain.)

Falsifiable: Let the domain be the natural numbers. Interpret 0 as zero, s as the successor function, but let *plus* be the subtraction function. Then, for every x, $x - 0 = x$ is true, and for every x and y, $(x+1) - y = (x - y) + 1$. However, the right hand side of the implication says that $0 = 4$, which is false.

Note that if we had written the second assumption as $\forall x(\forall y(plus(x, s(y)) \doteq s(plus(x, y)))))$ then this formula would not be falsifiable. Why?

3. Questions of Encoding I:

   Given the predicates defined in class for the beer-drinkers database example, provide existential formulas equivalent to the following queries:

   (a) What drinkers drink at least one beer served by the Hi-brow?

   $$\{person | (\exists beer(\exists price(serves(\mathsf{hibrow}, beer, price)) \wedge likes(person, beer))\}$$

   (b) What drinkers drink all the beers served by the Hi-brow?
   The core of the query is:

   $$\{person | (\forall beer(\exists price(serves(\mathsf{hibrow}, beer, price)) \Rightarrow likes(likes, beer))\}$$

   But if there are no beers served by the Hi-Brow, then the left of the implication is true for every value of the variable *beer*, making the implication true for every value, and the universal, then, true no matter what value we put in for *person*. So we need to either constrain the value of *person*, as in:

   $$\begin{aligned}\{person | \quad (\quad & (\exists beer(likes(person, beer)) \vee \exists bar(frequents(person, bar))) \wedge \\ & \forall beer(\exists price(serves(\mathsf{hibrow}, beer, price)) \Rightarrow likes(likes, beer))\}\end{aligned}$$

   Or we need to insure that the Hi-Brow actually serves some beers, as in:

   $$\begin{aligned}\{person | \quad (\quad & \exists beer(\exists price(serves(\mathsf{hibrow}, beer, price))) \wedge \\ & \forall beer(\exists price(serves(\mathsf{hibrow}, beer, price)) \Rightarrow likes(likes, beer))\}\end{aligned}$$

   (c) Who likes all the beers Hodas likes?
   As above the core is:

   $$\{person | (\forall beer(likes(\mathsf{hodas}, beer) \Rightarrow likes(person, beer, price))\}$$

   But similar constraints are needed, in case Hodas doesnt like any beers. So it should look like either:

   $$\begin{aligned}\{person | \quad (\quad & (\exists beer(likes(person, beer)) \vee \exists bar(frequents(person, bar))) \wedge \\ & \forall beer(likes(\mathsf{hodas}, beer) \Rightarrow likes(person, beer, price))\}\end{aligned}$$

   or

   $$\begin{aligned}\{person | \quad (\quad & \exists beer(likes(\mathsf{hodas}, beer)) \wedge \\ & \forall beer(likes(\mathsf{hodas}, beer) \Rightarrow likes(person, beer, price))\}\end{aligned}$$

(d) What bars does Hodas go to that serve all the beers he likes for less than \$3.00?

$$
\begin{aligned}
\{bar| \quad (\quad & frequents(\textsf{hodas}, bar) \wedge \\
& \forall beer(likes(\textsf{hodas}, beer) \Rightarrow \\
& \qquad (\exists price(serves(bar, beer, price) \wedge price < \$3.00)) \\
)\}
\end{aligned}
$$

(e) What bars are frequented by the drinkers who like all the beers that Hodas likes? I interpret this to mean that each bar in the answer is frequented by at least one such drinker.

$$
\begin{aligned}
\{bar|(\exists person \quad (\quad & frequents(person, bar) \wedge \\
& \forall beer(likes(\textsf{hodas}, beer) \Rightarrow \\
& \qquad likes(person, beer)) \\
))\}
\end{aligned}
$$

4. Questions of Encoding II:

Give first order encodings of each of the following facts:

(a) "Every fungus is either a mushroom or a toadstool."

$$\forall x(fungus(x) \Rightarrow (mushroom(x) \vee toadstool(x)))$$

(b) "Every boletus is a fungus."

$$\forall x(boletus(x) \Rightarrow fungus(x))$$

(c) "Toadstools are poisonous, as are peach pits."

$$\forall x(toadstool(x) \Rightarrow poisonous(x)) \wedge \forall x(peachpit(x) \Rightarrow poisonous(x))$$

or, alternatively:

$$\forall x((toadstool(x) \vee peachpit(x)) \Rightarrow poisonous(x))$$

(Because, $((A \vee B) \Rightarrow C)$ is equivalent to $(A \Rightarrow C) \wedge (B \Rightarrow C)$.)

(d) "A boletus is not a mushroom."

(e) "This thing is a boletus."

$$boletus(\textsf{thisThing})$$

(I also accepted $\forall x(thisThing(x) \Rightarrow boletus(x))$.)