Harvey Mudd College
Computer Science 80
Logic for Computer Science
Spring Semester 2002

Homework 7 – Predicate Logic: Proof Theory
Sample Solution

1. Give Natural Deduction proofs of each of the following formulas (brackets are used in place of parentheses in some formulas to make the structure clearer):

   (a) $\exists x[\forall y(p(x,y))] \Rightarrow \forall y[\exists x(p(x,y))]$

$$\dfrac{\dfrac{[\exists x(\forall y(p(x,y)))]^1 \quad \dfrac{\dfrac{\dfrac{[\forall y(p(x,y))]^2}{p(x,y)}\,\forall_E}{\exists x(p(x,y))}\,\exists_I}{\forall y(\exists x(p(x,y)))}\,\forall_I}{\forall y(\exists x(p(x,y)))}\,\exists_E 2}{\exists x(\forall y(p(x,y))) \Rightarrow \forall y(\exists x(p(x,y)))}\,\Rightarrow_I 1$$

   There are two crucial things to note in this proof:
   First, that in order to comply with the proviso for the $\exists_E$, the variable x must be re-bound (by the $\exists_I$ rule) above the use of $\exists_E$. (The use of $\forall_I$ could be moved below the $\exists_I$ rule since that rule puts no restrictions on y.)
   The more important thing to note is that this is how $\exists_E$ is used. A large number of students submitted solutions with the following non-proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\exists x(\forall y(p(x,y)))]^1}{\forall y(p(x,y))}\,\exists_E^*}{p(x,y)}\,\forall_E}{\exists x(p(x,y))}\,\exists_I}{\forall y(\exists x(p(x,y)))}\,\forall_I}{\exists x(\forall y(p(x,y))) \Rightarrow \forall y(\exists x(p(x,y)))}\,\Rightarrow_I 1$$

   The topmost rule here is NOT $\exists_E$. It is basically $\forall_E$ being misapplied to an existential. If this were the rule for eliminating existential quantifiers, all manner of invalid formulas could be proven. In particular, you would have proofs like:

$$\dfrac{\dfrac{\dfrac{[\exists x p(x)]^1}{p(x)}\,\exists_E^*}{\forall x p(x)}\,\forall_I}{\exists x p(x) \Rightarrow \forall x p(x)}\,\Rightarrow_I 1$$

(b) $\exists x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x(p(x)) \Rightarrow \exists x(q(x))]$

$$
\dfrac{
  \dfrac{
    [\exists x(p(x) \Rightarrow q(x))]^1 \qquad
    \dfrac{
      \dfrac{
        \dfrac{[\forall p(x)]^2}{p(x)}\ \forall_E \qquad [p(x) \Rightarrow q(x)]^3
      }{q(x)}\ \Rightarrow_E
    }{\exists x(q(x))}\ \exists_I
  }{
    \dfrac{\exists x(q(x))}{\forall x(p(x)) \Rightarrow \exists x(q(x))}\ \Rightarrow_I 2
  }\ \exists_E 3
}{\exists x(p(x) \Rightarrow q(x)) \Rightarrow (\forall x(p(x)) \Rightarrow \exists x(q(x)))}\ \Rightarrow_I 1
$$

Again, note the rebinding of x before the $\exists_E$ rule is reached.

(c) Prove that any boletus is poisonous. That is, from open assumptions:

- $\forall x(f(x) \Rightarrow [m(x) \vee t(x)])$
- $\forall x(b(x) \Rightarrow f(x))$
- $\forall x([t(x) \vee pp(x)] \Rightarrow p(x))$
- $\forall x(b(x) \Rightarrow \neg m(x))$

prove the conclusion $\forall x(b(x) \Rightarrow p(x))$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\dfrac{\forall x(b(x) \Rightarrow f(x))}{b(x) \Rightarrow f(x)}\ \forall_E}{f(x)}\ \Rightarrow_E \qquad
          \dfrac{\forall x(f(x) \Rightarrow (m(x) \vee t(x)))}{f(x) \Rightarrow (m(x) \vee t(x))}\ \forall_E
        }{m(x) \vee t(x)}\ \Rightarrow_E \qquad
        \dfrac{
          [m(x)]^2 \qquad
          \dfrac{[b(x)]^1 \qquad \dfrac{\forall x(b(x) \Rightarrow \neg m(x))}{b(x) \Rightarrow \neg m(x)}\ \forall_E}{\neg m(x)}\ \Rightarrow_E
        }{\bot}\ \neg_E \qquad
        [t(x)]^2
      }{t(x)}\ \vee_E 2 \qquad
      \dfrac{\forall x(t(x) \Rightarrow p(x))}{t(x) \Rightarrow p(x)}\ \forall_E
    }{p(x)}\ \Rightarrow_E
  }{b(x) \Rightarrow p(x)}\ \Rightarrow_I 1
}{\forall x(b(x) \Rightarrow p(x))}\ \forall_I
$$

(d) (extra credit) $[\forall x(p(x)) \Rightarrow \exists x(q(x))] \Rightarrow \exists x[p(x) \Rightarrow q(x)]$

Not yet.

2. Give Gentzen Sequent Calculus proofs for each of the conclusions in the last problem, including the extra credit (which is not extra credit in this problem).

   (a) $\exists x[\forall y(p(x,y))] \Rightarrow \forall y[\exists x(p(x,y))]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(d,c) \longrightarrow p(d,c)}\,id}{\forall y(p(d,y)) \longrightarrow p(d,c)}\,\exists_R}{\forall y(p(d,y)) \longrightarrow \exists x(p(x,c))}\,\exists_R}{\exists x(\forall y(p(x,y))) \longrightarrow \exists x(p(x,c))}\,\exists_L}{\exists x(\forall y(p(x,y))) \longrightarrow \forall y(\exists x(p(x,y)))}\,\forall_R}{\longrightarrow \exists x(\forall y(p(x,y))) \Rightarrow \forall y(\exists x(p(x,y)))}\,\Rightarrow_R$$

   (b) $\exists x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x(p(x)) \Rightarrow \exists x(q(x))]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q(c), p(c) \longrightarrow q(c)}\,id \quad \dfrac{}{p(c) \longrightarrow p(c), q(c)}\,id}{p(c) \Rightarrow q(c), p(c) \longrightarrow q(c)}\,\Rightarrow_L}{p(c) \Rightarrow q(c), p(c) \longrightarrow \exists x(q(x))}\,\exists_R}{p(c) \Rightarrow q(c), \forall x(p(x)) \longrightarrow \exists x(q(x))}\,\forall_L}{p(x) \Rightarrow q(x)), \forall x(p(x)) \longrightarrow \exists x(q(x))}\,\exists_L}{\exists x(p(x) \Rightarrow q(x)) \longrightarrow \forall x(p(x)) \Rightarrow \exists x(q(x))}\,\Rightarrow_R}{\longrightarrow \exists x(p(x) \Rightarrow q(x)) \Rightarrow (\forall x(p(x)) \Rightarrow \exists x(q(x)))}\,\Rightarrow_R$$

   (c) Prove that any boletus is poisonous. That is, from open assumptions:

   - $\forall x(f(x) \Rightarrow [m(x) \vee t(x)])$
   - $\forall x(b(x) \Rightarrow f(x))$
   - $\forall x([t(x) \vee pp(x)] \Rightarrow p(x))$
   - $\forall x(b(x) \Rightarrow \neg m(x))$

   prove the conclusion $\forall x(b(x) \Rightarrow p(x))$

   Let Γ be the set of assumptions listed above. The proof is then as follows, where each of the uses of $\forall_R$ is being applied to one of the clauses from Γ, instantiating it to the desired term, a:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, b(a), p(a) \longrightarrow p(a)}\,id \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, b(a), m(a) \longrightarrow m(a), t(a)}\,id}{\Gamma, b(a), m(a), \neg m(a) \longrightarrow t(a)}\,\neg_L \quad \dfrac{}{\Gamma, b(a), m(a) \longrightarrow b(a), t(a)}\,id}{\Gamma, b(a), m(a), b(a) \Rightarrow \neg m(a) \longrightarrow t(a)}\,\Rightarrow_L}{\Gamma, b(a), m(a) \longrightarrow t(a)}\,\forall_E \quad \dfrac{}{\Gamma, b(a), t(a) \longrightarrow t(a)}\,id}{\Gamma, b(a), m(a) \vee t(a) \longrightarrow t(a)}\,\vee_E \quad \dfrac{\dfrac{\dfrac{}{\Gamma, b(a), f(a) \longrightarrow f(a)}\,id \quad \dfrac{}{\Gamma, b(a) \longrightarrow b(a}}{\Gamma, b(a), b(a) \Rightarrow f(a) \longrightarrow f(a)}}{\Gamma, b(a) \longrightarrow f(a)}\,\forall_E}{\Gamma, b(a), f(a) \Rightarrow (m(a) \vee t(a)) \longrightarrow t(a)}\,\Rightarrow_L}{\Gamma, b(a) \longrightarrow t(a)}\,\forall_L}{\Gamma, b(a), t(a) \Rightarrow p(a) \longrightarrow p(a)}\,\Rightarrow_L}{\Gamma, b(a) \longrightarrow p(a)}\,\forall_L}{\Gamma \longrightarrow b(a) \Rightarrow p(a)}}{\Gamma \longrightarrow \forall x(b(x) \Rightarrow p(x))}\,\forall_R$$

(d) $(\forall x(p(x)) \Rightarrow \exists x(q(x))) \Rightarrow \exists x(p(x) \Rightarrow q(x))$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{q(c), p(c) \longrightarrow q(c)}\ id
}{q(c) \longrightarrow p(c) \Rightarrow q(c)}\ \Rightarrow_R
}{q(c) \longrightarrow \exists x(p(x) \Rightarrow q(x))}\ \exists_R
}{\exists x(q(x)) \longrightarrow \exists x(p(x) \Rightarrow q(x))}\ \exists_L
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{p(c) \longrightarrow q(c), p(c)}\ id
}{\longrightarrow p(c) \Rightarrow q(c), p(c)}\ \exists_R
}{\longrightarrow \exists x(p(x) \Rightarrow q(x)), p(c)}\ \exists_R
}{\longrightarrow \exists x(p(x) \Rightarrow q(x)), \forall x(p(x))}\ \forall_R
}{
\dfrac{(\forall x(p(x)) \Rightarrow \exists x(q(x))) \longrightarrow \exists x(p(x) \Rightarrow q(x))}{\longrightarrow (\forall x(p(x)) \Rightarrow \exists x(q(x))) \Rightarrow \exists x(p(x) \Rightarrow q(x))}\ \Rightarrow_R
}\ \Rightarrow_L
$$

3. Give resolution refutation proofs of each of the formulas in the last problem (i.e. show that each is a consequence of an empty Γ by negating just the one formula, converting it to PCNF, and performing resolution on the clauses that arise from it.)

   (a) $\exists x[\forall y(p(x,y))] \Rightarrow \forall y[\exists x(p(x,y))]$

$$
\begin{aligned}
\neg(\exists x[\forall y(p(x,y))] \Rightarrow \forall y[\exists x(p(x,y))]) &\longrightarrow \neg(\neg\exists x[\forall y(p(x,y))] \vee \forall y[\exists x(p(x,y))]) \\
&\longrightarrow \neg\neg\exists x[\forall y(p(x,y))] \wedge \neg\forall y[\exists x(p(x,y))] \\
&\longrightarrow \exists x[\forall y(p(x,y))] \wedge \exists y[\forall x(\neg p(x,y))] \\
&\longrightarrow \forall y(p(sk_1,y)) \wedge \forall x(\neg p(x,sk_2)) \\
&\longrightarrow p(sk_1,y) \wedge \neg p(x,sk_2)
\end{aligned}
$$

   (b) $\exists x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x(p(x)) \Rightarrow \exists x(q(x))]$

$$
\begin{aligned}
\neg(\exists x[p(x) \Rightarrow q(x)] \Rightarrow [\forall x(p(x)) \Rightarrow \exists x(q(x))]) &\longrightarrow \neg(\neg\exists x[\neg p(x) \vee q(x)] \vee [\neg\forall y(p(y)) \vee \exists z(q(z))]) \\
&\longrightarrow \neg\neg\exists x[\neg p(x) \vee q(x)] \wedge \neg[\neg\forall y(p(y)) \vee \exists z(q(z))]) \\
&\longrightarrow \exists x[\neg p(x) \vee q(x)] \wedge \neg\neg\forall y(p(y)) \wedge \neg\exists z(q(z))]) \\
&\longrightarrow \exists x[\neg p(x) \vee q(x)] \wedge \forall y(p(y)) \wedge \forall z(\neg q(z))]) \\
&\longrightarrow [\neg p(sk_1) \vee q(sk_1)] \wedge p(y) \wedge \neg q(z)
\end{aligned}
$$

(c) Prove that any boletus is poisonous. That is, from open assumptions:

- $\forall x(f(x) \Rightarrow [m(x) \vee t(x)])$
- $\forall x(b(x) \Rightarrow f(x))$
- $\forall x([t(x) \vee pp(x)] \Rightarrow p(x))$
- $\forall x(b(x) \Rightarrow \neg m(x))$

prove the conclusion $\forall x(b(x) \Rightarrow p(x))$

$$
\begin{aligned}
\forall x(f(x) \Rightarrow [m(x) \vee t(x)]) &\longrightarrow \neg f(x) \vee m(x) \vee t(x) \\
\forall x(b(x) \Rightarrow f(x)) &\longrightarrow \forall y(b(y) \Rightarrow f(y)) \longrightarrow \neg b(y) \vee f(y) \\
\forall x([t(x) \vee pp(x)] \Rightarrow p(x)) &\longrightarrow \forall z(\neg[t(z) \vee pp(z)] \vee p(z)) \longrightarrow [\neg t(z) \wedge \neg pp(z)] \vee p(z) \\
&\longrightarrow (\neg t(z) \vee p(z)) \wedge (\neg pp(z) \vee p(z)) \\
\forall w(b(w) \Rightarrow \neg m(w)) &\longrightarrow \neg b(w) \vee \neg m(w) \\
\neg(\forall x(b(x) \Rightarrow p(x)) &\longrightarrow \neg(\forall v(\neg b(v) \vee p(v))) \longrightarrow \exists v(\neg\neg b(v) \wedge \neg p(v)) \\
&\longrightarrow b(sk_1) \wedge \neg p(sk_1)
\end{aligned}
$$

(d) $(\forall x(p(x)) \Rightarrow \exists x(q(x))) \Rightarrow \exists x(p(x) \Rightarrow q(x))$

$$
\begin{aligned}
\neg[(\forall x(p(x)) \Rightarrow \exists x(q(x))) \Rightarrow \exists x(p(x) \Rightarrow q(x))] &\longrightarrow \neg[\neg(\neg\forall x(p(x)) \vee \exists y(q(y))) \vee \exists z(\neg p(z) \vee q(z))] \\
&\longrightarrow \neg\neg(\neg\forall x(p(x)) \vee \exists y(q(y))) \wedge \forall z\neg(\neg p(z) \vee q(z))] \\
&\longrightarrow (\exists x(\neg p(x)) \vee \exists y(q(y))) \wedge \forall z(\neg\neg p(z) \wedge \neg q(z)) \\
&\longrightarrow (\neg p(sk_1) \vee q(sk_2)) \wedge p(z) \wedge \neg q(z)
\end{aligned}
$$