Logical Consequence

The flip-side of Theorem 2.5.12 is a bit suprising at first:

Theorem: For any formula $\Phi \in PROP$, $\bot \models \Phi$.

Proof.

Logical Consequence

Definition: Given two formulas $\Phi, \Psi \in PROP$, we say that Φ *is logically stronger than* Ψ if $\Phi \models \Psi$ but $\Psi \not\models \Phi$. Or, equivalently, if $\models \Phi \Rightarrow \Psi$ but $\not\models \Psi \Rightarrow \Phi$

Theories

Definition: Given a set of formulas $U = \{\Phi_1, \ldots, \Phi_n\} \subseteq PROP$, the set of formulas $\mathcal{T}(U) = \{\Psi | U \models \Psi\}$ is called *the theory of* U. Each element of $\mathcal{T}(U)$ is called a *theorem*.

Random Sidebar

What is the simplest tautology?

What is the next simplest tautology?

What is the simplest tautology that has no uses of logical constants (i.e. $\top$ or $\bot$)?

Logical Equivalence

As with $\Rightarrow$ and $\models$, there is a meta-level partner to the operator $\equiv$.

Definition: Given two formulas $\Phi, \Psi \in PROP$, if for all valuations v, $\hat{v}(\Phi) = \hat{v}(\Psi)$, we say that Φ is *logically equivalent to* Ψ, written $\Phi \leftrightarrow \Psi$.

Theorem (2.44)**:** Given two formulas, $\Phi, \Psi \in PROP$, $\Phi \leftrightarrow \Psi$ iff $\models (\Phi \equiv \Psi)$.

Corollary: For two formulas, $\Phi, \Psi \in PROP$, $\Phi \leftrightarrow \Psi$ iff:

- $\models (\Phi \Rightarrow \Psi)$ and $\models (\Psi \Rightarrow \Phi)$
- $\models ((\Phi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \Phi))$
- $\Phi \models \Psi$ and $\Psi \models \Phi$

Substitution

Definition: Given $\Phi, \Psi \in PROP$ such that Φ is a subformula of Ψ, then, for any formula $\Xi \in PROP$, the *substitution of* Ξ *for* Φ *in* Ψ, written $\Psi[\Phi \leftarrow \Xi]$, is the formula whose formation tree is obtained by replacing the formation subtree of Φ in the formation tree of Ψ with the formation tree of Ξ.

This slightly cleaned up version of the definition given in the book is still not well defined. Why?

The Substitution Theorem

Theorem (2.4.7)**:** If formulas $\Phi, \Psi, \Xi \in PROP$ are such that Φ is a subformula of Ψ, and $\Xi \leftrightarrow \Phi$, then $\Psi \leftrightarrow \Psi[\Phi \leftarrow \Xi]$.

Proof.

Some Useful Equivalences

Laws of 0 and 1:	$A \vee \top \leftrightarrow \top$	$A \wedge \top \leftrightarrow A$
	$A \vee \bot \leftrightarrow A$	$A \wedge \bot \leftrightarrow \bot$
	$A \Rightarrow \top \leftrightarrow \top$	$\top \Rightarrow A \leftrightarrow A$
	$A \Rightarrow \bot \leftrightarrow \neg A$	$\bot \Rightarrow A \leftrightarrow \top$
More 0 and 1:	$A \vee \neg A \leftrightarrow \top$	$A \wedge \neg A \leftrightarrow \bot$
	$\top \leftrightarrow A \Rightarrow A$	
	$\top \leftrightarrow A \equiv A$	$\bot \leftrightarrow A \not\equiv A$
	$\neg A \leftrightarrow A \uparrow A$	$\neg A \leftrightarrow A \downarrow A$
Double Negation:	$A \leftrightarrow \neg\neg A$	
Idempotency:	$A \leftrightarrow A \wedge A$	$A \leftrightarrow A \vee A$
Commutativity:	$A \vee B \leftrightarrow B \vee A$	$A \wedge B \leftrightarrow B \wedge A$
	$A \equiv B \leftrightarrow B \equiv A$	$A \not\equiv B \leftrightarrow B \not\equiv A$
	$A \uparrow B \leftrightarrow B \uparrow A$	$A \downarrow B \leftrightarrow B \downarrow A$
Contrapositive:	$A \Rightarrow B \leftrightarrow \neg B \Rightarrow \neg A$	
Associativity:	$A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$	$A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$
	$A \equiv (B \equiv C) \leftrightarrow (A \equiv B) \equiv C$	$A \not\equiv (B \not\equiv C) \leftrightarrow (A \not\equiv B) \not\equiv C$
Distributivity:	$A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$	$A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)$
Absorption:	$A \wedge (A \vee B) \leftrightarrow A$	$A \vee (A \wedge B) \leftrightarrow A$
DeMorgan's Laws:	$A \Rightarrow B \leftrightarrow \neg A \vee B$	$A \Rightarrow B \leftrightarrow \neg(A \wedge \neg B)$
	$A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$	$A \wedge B \leftrightarrow \neg(\neg A \vee \neg B)$
	$A \vee B \leftrightarrow \neg A \Rightarrow B$	$A \wedge B \leftrightarrow \neg(A \Rightarrow \neg B)$
Other Defs:	$A \equiv B \leftrightarrow (A \Rightarrow B) \wedge (B \Rightarrow A)$	
	$A \Rightarrow B \leftrightarrow A \equiv (A \wedge B)$	$A \Rightarrow B \leftrightarrow B \equiv (A \vee B)$
	$A \wedge B \leftrightarrow (A \equiv B) \equiv (A \vee B)$	$A \equiv B \leftrightarrow (A \vee B) \Rightarrow (A \wedge B)$