Intuitionistic (Constructive) Logic

Up till now, we have focused on what is now known as *classical logic*. For 2000 years it was the only kind of logic there was.

In the late 1800's, however, a school of Philosophy and Mathematics grew up that took issue with an aspect of logic: the possibility of proving the existence of an object without actually showing the object.

Consider the proof of the following conjecture:

Conjecture: Does there exists a pair of irrational numbers a and b such that a^b is rational?

Proof.

Intuitionistic (Constructive) Logic

The school of *constructivism* has its roots in writings of *Emanuel Kant.*

Constructive logics have the *Disjunctive Property* and the *Existential Property.*

Definition: A logic, $\mathcal{L}$, has the *disjunctive property* iff, whenever $\vdash_{\mathcal{L}} \phi \vee \psi$, either $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$.

Definition: A quantificational logic, $\mathcal{L}$, has the *existential property* iff, whenever $\vdash_{\mathcal{L}} \exists x.(\phi(x))$, then there is a t such that $\vdash_{\mathcal{L}} \phi(t)$.

These ideas really come to fruition in the work of the Dutch Mathematician *L. E. J. Brouwer*, who coined the term *intuitionism.*

At the level of proofs the problem comes down to any use of a mechanism equivalent to the *law of the excluded middle*: $A \vee \neg A$.

Intuitionistic (Constructive) Logic

Just because we have exhibited a non-constructive proof, does that mean that there is a difference between constructive and classical logic? Perhaps we can still prove all the same things.

The answer, though, is no. A trivial example is the formula $a \vee \neg a$ which is classically true, and provable, but not intuitionistically true, or provable.

Now, if some things are not provable, then there is a difference in the consequence relation $\models$. But since we have a theorem that mates consequence at the meta level and implication at the object level, there must be some change to the meaning of implication.

The Problem With Implication

In classical logic, the truth of the law of the excluded middle leads to the semantic blurring of implication: the fact that $(A \Rightarrow B) \leftrightarrow (\neg A \vee B)$. This equivalence does not hold in intuitionistic systems.

Though we have previously convinced ourselves that this equivalence made sense, it has some seemingly non-sensical consequences, which are barred in intuitionistic logic. For example, in classical logic we have the following chain of equivalences:

$$
\begin{aligned}
(A \Rightarrow B) \vee C &\leftrightarrow (\neg A \vee B) \vee C \\
&\leftrightarrow \neg A \vee (B \vee C) \\
&\leftrightarrow \neg A \vee (C \vee B) \\
&\leftrightarrow (\neg A \vee C) \vee B \\
&\leftrightarrow (A \Rightarrow C) \vee B
\end{aligned}
$$

Thus the assumption in an implication is seen to bleed out of scope (in the programming sense) and apply to some other conclusion.

Pierce's Formula

The simplest purely implicational formula that exhibits this behavior is one called Pierce's Formula:

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

This featured on the last homeworks in which you showed it was a tautology. I pointed out, however, that I suspected it would be very hard for you to give it a "reasonable" reading. The reason is that it is a demonstration of the way in which classical logic fails to properly model our notion of implication.

Classical Natural Deduction

As shown so far, the system is *constructive*.

In order to be able to do proofs by contradiction, which do not produce a witness, and thereby to make the system *classical*, you can either:

- Allow axioms of the form $\phi \vee \neg\phi$ (explicit invocations of the law of the excluded middle).
- Or, add the double negation rule to the system:

$$\frac{\neg\neg\phi}{\phi}$$

- Or, add the rule *reductio ad absurdum*:

$$\begin{array}{c} \neg\not\phi \\ \vdots \\ \dfrac{\bot}{\phi}\ RAA \end{array}$$

This ability to distinguish intuitionistic from classical reasoning based just on the availability of certain rules, further distinguishes natural deduction from Hilbert systems, in which it is harder to see how to make the separation.

Classical Natural Deduction

Note that while it is quite similar, this last, RAA, is not the same as the $\neg_E$ rule:

$$\dfrac{\begin{array}{c}\not\phi\\ \vdots\\ \bot\end{array}}{\neg\phi}\,\neg E \qquad\qquad \dfrac{\begin{array}{c}\neg\not\phi\\ \vdots\\ \bot\end{array}}{\phi}\,RAA$$

What are the readings of the two rules?

Some Classical Natural Deduction Proofs

$$\frac{\vdots}{\phi \vee \neg\phi}$$

$$\frac{}{((p \Rightarrow q) \Rightarrow p) \Rightarrow p}$$