Free Variables of a Term

Definition: The free variables of a well-formed term are given by the function $FV_{TERMS} : TERMS \rightarrow \wp(\mathcal{X})$ defined recursively as follows:

- $FV(a) = \emptyset$ for $a \in \mathcal{A}$
- $FV(x) = \{x\}$ for $x \in \mathcal{X}$
- $FV(f(t_1, \ldots, t_n)) = FV(t_1) \cup \ldots \cup FV(t_n)$ for $f \in \mathcal{F}$ with $arity_{\mathcal{F}}(f) = n$

Note that there is no such thing as the bound variables of a term. All variables in a term are free (which is a simple, non-recursive definition of FV_{TERMS}).

Free Variables of a Formula

Definition: The free variables of a well-formed formula are given by the function $FV_{FORM} : FORM \to \wp(\mathcal{X})$ defined recursively as follows:

- $FV(\bot) = FV(\top) = \emptyset$
- $FV(p(t_1, \ldots, t_n)) = FV(t_1) \cup \ldots \cup FV(t_n)$ for $p \in \mathcal{P}$ with $arity_{\mathcal{P}}(p) = n$
- $FV((\neg A)) = FV(A)$
- $FV((A_1 \bullet A_2)) = FV(A_1) \cup FV(A_2)$ for $\bullet \in \{\wedge, \vee, \Rightarrow, \equiv\}$
- $FV(\forall x(A)) = FV(A) - \{x\}$
- $FV(\exists x(A)) = FV(A) - \{x\}$

Bound Variables of a Formula

Definition: The bound variables of a well-formed formula are given by the function $BV_{FORM} : FORM \rightarrow \wp(\mathcal{X})$ defined recursively as follows:

- $BV(\bot) = BV(\top) = \emptyset$
- $BV(p(t_1, \ldots, t_n)) = \emptyset$
  for $p \in \mathcal{P}$ with $arity_{\mathcal{P}}(p) = n$
- $BV((\neg A)) = BV(A)$
- $BV((A_1 \bullet A_2)) = BV(A_1) \cup BV(A_2)$
  for $\bullet \in \{\wedge, \vee, \Rightarrow, \equiv\}$
- $BV(\forall x(A)) = BV(A) \cup \{x\}$
- $BV(\exists x(A)) = BV(A) \cup \{x\}$

Note: The sets $FV(A)$ and $BV(A)$ are not necessarily disjoint.

Substitution of a Term for a Variable

Definition: Let s and t be well-formed terms, and x a variable. Then the result of substituting the term t for the variable x in s, written $s[x := t]$ is defined recursively as follows:

- $a[x := t] = a$, for $a \in \mathcal{A}$
- $y[x := t] = t$, for $(y = x) \in \mathcal{X}$
- $y[x := t] = y$, for $(y \neq x) \in \mathcal{X}$
- $f(t_1, \ldots, t_n)[x := t] = f(t_1[x := t], \ldots, t_n[x := t])$, for $f \in \mathcal{F}$ with $arity_{\mathcal{F}}(f) = n$

Substitution of a Term for a Variable

Definition: Given $A \in FORM$, $t \in TERM$, and $x \in X$. Then the result of substituting the term t for the variable x in A, written $A[x := t]$ is defined recursively as follows:

- $\bot[x := t] = \bot$, and $\top[x := t] = \top$
- $p(t_1, \ldots, t_n)[x := t] = p(t_1[x := t], \ldots, t_n[x := t])$ for $p \in \mathcal{P}$ with $arity_{\mathcal{P}}(p) = n$
- $(\neg A)[x := t] = (\neg A[x := t])$
- $(A_1 \bullet A_2)[x := t] = (A_1[x := t] \bullet A_2[x := t])$ for $\bullet \in \{\wedge, \vee, \Rightarrow, \equiv\}$
- $\mathcal{Q}y(A)[x := t] = \mathcal{Q}y(A)$ if $y = x$, for $\mathcal{Q} \in \{\forall, \exists\}$
- $\mathcal{Q}y(A)[x := t] = \mathcal{Q}y(A[x := t])$ if $y \neq x$, for $\mathcal{Q} \in \{\forall, \exists\}$

Substitution of a Term for a Variable

There is a problem with that last definition (which is essentially the one given in Ben-Ari). What is it?

A Term *Free* for a Variable

Definition: Given $A \in FORM$, $t \in TERM$, and $x \in X$. Then we say *t is free for x in A* (that is, t is okay to substitute for x in A) if:

- $A \in ATOM$, or
- $A = (\neg A_1)$, and t is free for x in A_1, or
- $A = (A_1 \bullet A_2)$ for $\bullet \in \{\wedge, \vee, \Rightarrow, \equiv\}$, and t is free for x in A_1 and A_2, or
- $A = \mathcal{Q}y(A_1)$, for $\mathcal{Q} \in \{\forall, \exists\}$, and either
  - $x = y$, or
  - $x \neq y$, and $y \notin FV(t)$, and t is free for x in A_1

Substitution of a Term for a Variable, Redux

Definition: Given $A \in FORM$, $t \in TERM$, and $x \in X$. Then the result of substituting the term t for the variable x in A, written $A[x := t]$ is defined recursively as follows:

- $\bot[x := t] = \bot$, and $\top[x := t] = \top$
- $p(t_1, \ldots, t_n)[x := t] = p(t_1[x := t], \ldots, t_n[x := t])$ for $p \in \mathcal{P}$ with $arity_{\mathcal{P}}(p) = n$
- $(\neg A)[x := t] = (\neg A[x := t])$
- $(A_1 \bullet A_2)[x := t] = (A_1[x := t] \bullet A_2[x := t])$ for $\bullet \in \{\wedge, \vee, \Rightarrow, \equiv\}$
- $\mathcal{Q}y(A)[x := t] = \mathcal{Q}y(A)$, for $\mathcal{Q} \in \{\forall, \exists\}$, if $y = x$.
- $\mathcal{Q}y(A)[x := t] = \mathcal{Q}y(A[x := t])$ for $\mathcal{Q} \in \{\forall, \exists\}$, if $y \neq x$, **and** t is free for x in A.

  Otherwise, $\mathcal{Q}y(A)[x := t] = \mathcal{Q}z(A[y := z])[x := t]$, where z is a fresh variable.