Algorithms
Computer Science 140 & Mathematics 168
Instructor: B. Thom
Fall 2003
Homework 12b
Due on 12/9/03 (in class)

1. *This homework set is quite challenging. Please start early and please come to office hours.* In this homework set you will prove that several problems are NP-complete. Recall that proving NP-completeness has several parts: First you will need to prove that the problem is in NP (which is usually easy). Next, you must describe a reduction from a known NP-complete problem to this problem and explain why the reduction can be performed in polynomial time. Finally, you must show that the answer to the problem on the left-hand side of the reduction operator $\leq_p$ is true if and only if the answer to the problem on the right-hand side of the reduction operator is true. Be sure to write out your NP-completeness proofs carefully and precisely!

2. [**20 Points**] **More on ILP!** In class we proved that Integer Linear Programming (ILP) is NP-complete. Our reduction was from 3SAT. In this problem you will prove that ILP is NP-complete using two different reductions. You do not need to show that ILP is in NP (we already argued that in class). You need only give a reduction, prove its correctness (remember, this is an "if and only if" proof), and explain why the reduction can be done in polynomial time.

   (a) Show that ILP is NP-complete using a reduction from Vertex Cover.

   (b) Show that ILP is NP-complete using a reduction from Subset Sum.

3. [**20 Points**] **SPAM is NP-Complete!** It's true! Here's the story: My-I'm-Soft has just released it's new product "Portholes 1987" and Gill Bates has decided to celebrate by throwing a big party for his employees. Unfortunately, many of the employees at My-I'm-Soft do not get along with one another[1]. Gill would like to invite as many employees to the party as possible such that no two partygoers dislike one another. This is known as the **S**electing **P**artygoers **A**t **M**y-I'm-Soft (**SPAM**) Problem.

   Given is a set $E = \{e_1, \ldots, e_n\}$ of n employees and a set H which consists of pairs of employees (e_i, e_j) such that employees e_i and e_j do not get along. In the optimization problem, we wish to find the largest subset $S \subseteq E$ such that for every pair of employees $e_k, e_\ell \in S$, $(e_k, e_\ell) \notin H$.

   (a) Formulate the SPAM problem as a decision problem.

   (b) Prove that the SPAM decision problem is NP-complete. (*Note:* Use one of 3SAT, Vertex Cover, Independent Set, or Clique in your reduction.)

[1] The famous social satirist Tom Lehrer once said "There are people in this world who do not love one another and I **HATE** people like that!"

4. **[25 Points] The Kernel!** Consider a directed graph $G = (V, E)$. A *kernel* for graph G is a subset K of the vertices such that:

   (a) Every vertex in V is either in the kernel K or has an incoming edge from some vertex in the kernel. (Formally, "$\forall v \in V$, $v \in K$ or $\exists u \in K$ s.t. $(u, v) \in E$.")

   (b) There do not exist two vertices in the kernel K with a directed edge between them. (Formally, "$\forall u, v \in K$, $(u, v) \notin E$ and $(v, u) \notin E$.")

   Observe that the graph below has a kernel. (A set containing just one of the two vertices is a kernel.)

   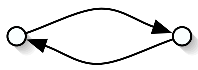

   Figure 1: This directed cycle has a kernel.

   On the other hand, the graph in Figure 2 has no kernel. (Take a look at the definition of the kernel to make sure you see why.)

   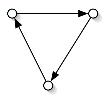

   Figure 2: This directed cycle has no kernel.

   The **Kernel Problem** is the following: Given a directed graph $G = (V, E)$, does G have a kernel? Prove that the Kernel Problem is NP-complete. (Hint: Perform a reduction from 3SAT and use the graphs in Figures 1 and 2 as the gadgets.)