

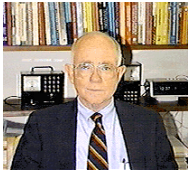
Adalines

Primitive Artificial Neurons

Sweet Adaline

- The Adaline (Adaptive Linear Neuron or Adaptive Linear Element) is a model similar to the Perceptron. There are several variations:
 - One has the threshold function similar to a perceptron.
 - Another uses a **pure linear** function with no threshold.

Adaline Inventors
Bernard Widrow and Marcian (Ted) Hoff



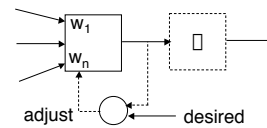
Bernard Widrow,
Professor Emeritus of E.E.,
Stanford University



Marcian Hoff
Co-inventor of Patent 3,821,715
*Microprocessor Concept and
Architecture*

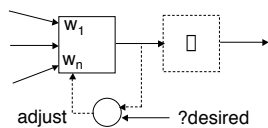
Adaline Training (1)

- With or without the threshold, the Adaline is **trained** based on the output of the linear function rather than the final output.



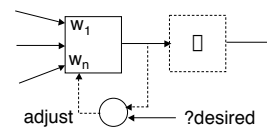
Adaline Training (2)

- The catch here is that we have to state the **desired** value in terms of the output of the linear part, rather than the output after the threshold.
- What is this for a classifier?



Adaline Training (2)

- A reasonable approach is to use a *nominal* value such as -0.5 as desired for a “no” classification and a +0.5 for a “yes” classification.



Adaline Training (3)

- The formula for Adaline weight updating is very similar to the Perceptron:

Add to the weights Δw where

$$\Delta w = \eta \delta [-1, x_1, x_2, \dots, x_n]$$

Adaline learning rule

only now the **error is not limited to 1, -1, 0 as before**; it can have a fractional value, since it is based on the output of the linear part of the device.

Adaline Training (4)

- The weight update formula for the Adaline will be justified eventually.
- One major difference from this vs. the Perceptron is that a learning rate of 1 won't generally be acceptable. It will need to be smaller, say 0.01. There is a theory that tells us how large we can make it.

Adaline Example

- We'll use the same example as before. But now we'll train on the output of the linear portion and target for +1 for a "yes" answer and -1 for a "no" answer.
 - (4, 5) +1
 - (6, 1) +1
 - (4, 1) -1
 - (1, 2) -1
- Try a learning rate of 0.01.

Adaline Training Example

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	1			
	(-1, 6, 1)	1			
	(-1, 4, 1)	-1			
	(-1, 1, 2)	-1			

Adaline Training Example

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	1	-1	2	
	(-1, 6, 1)	1			
	(-1, 4, 1)	-1			
	(-1, 1, 2)	-1			(0.0358, 0.727, -0.943)

We stopped (after 500 epochs) at (0.67, 0.21, 0.17)

weights	input	desired	actual	error
(0.67, 0.21, 0.17)	(-1, 4, 5)	1	1.02	-0.02
	(-1, 6, 1)	1	0.75	0.25
	(-1, 4, 1)	-1	0.33	-1.33
	(-1, 1, 2)	-1	-0.12	-0.88

With a threshold of 0.67, this would correctly separate the samples.

Adaline Convergence (2)

- The Adaline admits a more refined stopping criterion:

The **Mean-Squared Error (MSE)** is the average of the squares of the error taken over all samples. Squaring makes the measure insensitive to the sign of the error. It also provides certain analytic properties.
- This quantity ideally converges toward a specific minimum (which might never be exactly attained). The algorithm can be set to stop when the MSE reaches a desired value.

Alternate Rule Names

- Because the Adaline rule minimizes MSE, it is sometimes called the “**LMS rule**” [LMS = “least mean square”].
- The term “**Delta rule**” is also sometimes used, although this will be seen to be a rule for a more general class of networks.

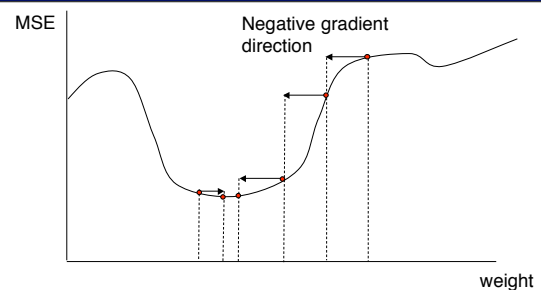
Minimizing Error

- Consider the task of finding the minimum of a function of one variable.
- One standard method for doing this, if the derivative of the function is known, is Newton’s method, which entails constructing a tangent line from a current estimate, then using the intersection of the line with the origin for the next.

Gradient Descent

- Gradient descent is another method for finding the minimum.
- It consists of computing the gradient of the function, then taking a small step in the direction of negative gradient, which hopefully corresponds to decreased function value, then repeating for the new value of the dependent variable.

Gradient Descent

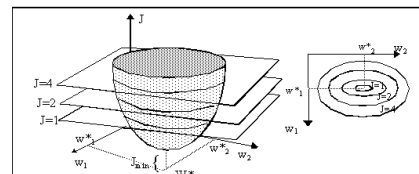


Gradient Descent

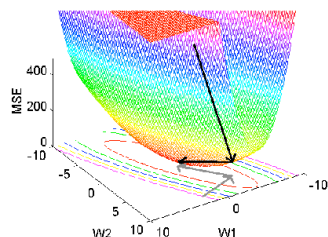
- A single dimension for weights (including bias) is atypical.
- So the previous diagram is mainly to enhance our intuition.
- For the general case, the gradient is a **vector** of gradient components, one for each weight (including bias).

Error vs. 2-D Weight Space

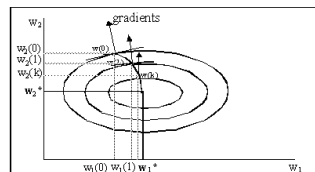
(one weight is threshold or -bias)



2-D Gradient Descent



2-D Gradient Descent



Computing Gradients

Note:
squared

- $MSE = J(w) = \frac{1}{n} \sum (\text{desired} - \text{actual})^2$
where \sum is over n samples.
- desired is a fixed constant for each sample.
- actual = $\sum w_j x_j$ (\sum over input lines)
- So $J(w) = \frac{1}{n} \sum (\text{desired} - \sum w_j x_j)^2$

On-Line Approximation to Gradient

- On-line means based on a single sample, rather than batch, which means using all samples
- $J \approx (d - \sum w_j x_j)^2$ (note: no outer sum) (d = desired)
- i^{th} gradient component = $\partial J / \partial w_i$
 $= \partial / \partial w_i (d - \sum w_j x_j)^2$
 $= 2 (d - \sum w_j x_j) \partial / \partial w_i (d - \sum w_j x_j) = -2 \sum x_j$
 $= \underbrace{2 (d - \sum w_j x_j)}_{\text{error, } e} \underbrace{-1}_{-x_i}$

Computing Gradients

- i^{th} gradient component = $-2 \sum x_j$
- However we want to move in the *direction of negative* gradient, tempered by the learning rate η , so:

Amount to *add* to weight is

$$\Delta w_i = 2 \eta \sum x_j$$

which we recognize as the LMS (Adaline) rule (2 can be folded into η).

Vector Version of the Analysis of Gradient Descent for Adaline

- $MSE = J(w) = E[(d - w^T x)^2]$ (E = expectation or mean averaged over samples)
 $= E[d^2 - 2d w^T x + w^T x x^T w]$
 $= E[d^2] - E[2d w^T x] + E[w^T x x^T w]$
 $= E[d^2] - 2w^T E[d x] + w^T E[x x^T] w$
 $= c - 2w^T h + w^T R w$, for appropriate const. c, h, R

Vector Analysis of Gradient Descent for Adaline

- $J(w) = c - 2w^T h + w^T R w$
where $c = E[d^2]$, $h = E[d x]$, $R = E[x x^T]$
- This is a **quadratic form** in w with coefficients derived from the data vectors x .
- R is called the (auto-)**correlation matrix**.

Standard Quadratic Form

- $J(w) = c + w^T b + (1/2)w^T A w$
where $c = E[d^2]$,
 $b = -2E[d x]$,
 $A = 2E[x x^T]$
- A is called the **Hessian** matrix. It is the matrix of 2nd partial derivatives of the surface.

Analytic Gradient

- $\nabla J(w) = \nabla (c + w^T b + (1/2)w^T A w) = b + A w$
- It can be shown that if J has a **minimum**, it will be at a point w^* where $\nabla J(w^*) = 0$,
i.e. $b + A w^* = 0$
i.e. $w^* = A^{-1} b$
- where $A = 2E[x x^T]$, $b = -2E[d x]$

Stable Points

- In general, $w^* = A^{-1} b$ is a stable point.
- It may correspond to a minimum, maximum, or saddle.

Convergence of Gradient Descent for Adaline

- $\Delta w = 2 \eta \nabla J(w)$
 $w(k+1) = w(k) + 2 \eta \nabla J(w(k))$ (at k^{th} step)
- $E[w(k+1)] = E[w(k)] + 2 \eta E[\nabla J(w(k))]$
... math ...
 $= (I - 2\eta R) E[w(k)] + 2\eta h$
- For convergence, the eigenvalues of the matrix $I - 2\eta R$ must be within the unit circle.

Convergence of Gradient Descent for Adaline

- If λ_i is an eigenvalue of R , convergence requires $|1 - 2 \eta \lambda_i| < 1$.
- which simplifies to $\eta < 1 / \lambda_i$ for all eigenvalues λ_i , in particular for the maximum one.

$$\eta < 1 / \lambda_{\max}$$

Bound on learning rate for convergence of Adaline training by gradient descent

NAS repeatedly uses $2 / \lambda_{\max}$, possible because it doesn't keep the factor of 2 in the learning rule.

Example (from NND)

Sample 1 $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{d}_1 = \begin{bmatrix} -1 \end{bmatrix}$ Sample 2 $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{d}_2 = \begin{bmatrix} 1 \end{bmatrix}$

$$\mathbf{R} = E[\mathbf{x}\mathbf{x}^T] = \frac{1}{2}\mathbf{x}_1\mathbf{x}_1^T + \frac{1}{2}\mathbf{x}_2\mathbf{x}_2^T$$

$$\mathbf{R} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda_1 = 1.0, \quad \lambda_2 = 0.0, \quad \lambda_3 = 2.0$$

$$\eta < \frac{1}{\lambda_{\max}} = \frac{1}{2.0} = 0.5$$

Training, 1st Epoch

Sample 1 $a(0) = \mathbf{W}(0)\mathbf{p}(0) = \mathbf{W}(0)\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0$

$$\delta(0) = d(0) - a(0) = d_1 - a(0) = -1 - 0 = -1$$

$$\mathbf{W}(1) = \mathbf{W}(0) + 2\delta(0)\mathbf{x}^T(0)$$

$$\mathbf{W}(1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + 2(0.2)(-1) \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} -0.4 & 0.4 & 0.4 \end{bmatrix}$$

2nd Epoch

Sample 2 $a(1) = \mathbf{W}(1)\mathbf{p}(1) = \mathbf{W}(1)\mathbf{p}_2 = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -0.4$

$$\delta(1) = d(1) - a(1) = d_2 - a(1) = 1 - (-0.4) = 1.4$$

$$\mathbf{W}(2) = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} + 2(0.2)(1.4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix}$$

3rd Epoch

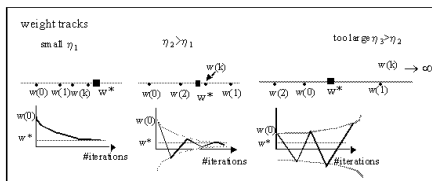
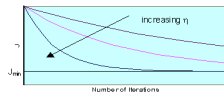
$$a(2) = \mathbf{W}(2)\mathbf{p}(2) = \mathbf{W}(2)\mathbf{p}_1 = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -0.64$$

$$\delta(2) = d(2) - a(2) = d_1 - a(2) = -1 - (-0.64) = -0.36$$

$$\mathbf{W}(3) = \mathbf{W}(2) + 2\delta(2)\mathbf{x}^T(2) = \begin{bmatrix} 1.104 & 0.016 & -0.016 \end{bmatrix}$$

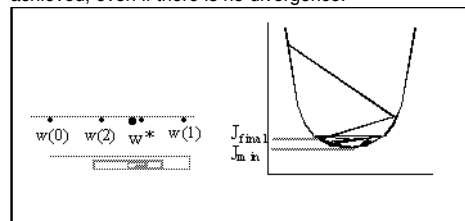
$$\mathbf{W}(\infty) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

"Learning Curve"



"Rattling"

If learning rate too high, minimum solution won't be achieved, even if there is no divergence.

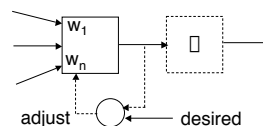


Solutions to Rattling

- Rule of thumb: Use a learning rate 0.1 times the theoretical maximum, or
- Gradually decrease the learning rate, e.g. according to a pre-set schedule, or
- Adaptively set the learning rate:
 - If the MSE is taking big steps, make it smaller.
 - If the MSE is taking small steps, make it larger.

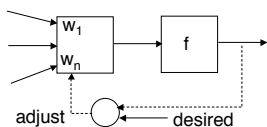
Generalizing the Adaline

- We have already discussed the problem of a threshold output in this picture



Generalizing the Adaline

- Suppose that we replace the threshold stage with a general function f and revert to expressing desired in terms of its output:



Generalizing the Adaline

- Consider again the derivation of the LMS rule:

$$J(w) = \frac{1}{2} (\text{desired} - f(\sum w_j x_j))^2 / n$$
- $J \approx (d - f(\sum w_j x_j))^2$ ($d = \text{desired}$)
- i^{th} gradient component = $\partial J / \partial w_i$

$$= \frac{\partial}{\partial w_i} (d - f(\sum w_j x_j))^2$$

$$= 2 (d - f(\sum w_j x_j)) \frac{\partial}{\partial w_i} (d - f(\sum w_j x_j))$$

$$= -2 \frac{\partial}{\partial w_i} f(\sum w_j x_j)$$

$$= -2 f'(\sum w_j x_j) \frac{\partial}{\partial w_i} \sum w_j x_j = -2 x_i f'(\sum w_j x_j)$$

Generalized LMS Rule (or Delta Rule)

- $\Delta w = 2 \eta x_i f'(\sum w_j x_j)$ assuming that f has a derivative f' .
- $\sum w_j x_j$ is often called the “**net**” value or “**activation**” value, and f the **activation function**.
- For the special case of f being the identity function, this reduces to the LMS rule we had before.

Generalized LMS Rule (or Delta Rule)

- Why worry about this generalization?

$$\Delta w = 2 \eta x_i f'(\sum w_j x_j)$$
- It will have a number important uses.

Use #1

- In the Adaline with threshold, we can't very well treat the model analytically, due to the fact that we have a non-continuous function at the output.
- But we can *approximate* the non-continuous function with a continuous one:

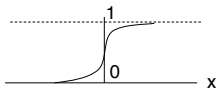


Sigmoids

- The “S” shape on the right of the previous slide is called a sigmoid curve.
- This is a generic term and there are several different analytic functions that behave this way.

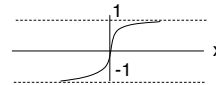
Logisitic Sigmoid

- Logisitic function (“logsig”—Matlab):
 $f(x) = 1/(1+\exp(-ax))$
- $f'(x) = f(x)(1-f(x))$



Hyperbolic Sigmoid

- Hyperbolic tangent function (“tansig”):
 $f(x) = \tanh(x)$
 $= (\exp(x) - \exp(-x)) / (\exp(x) + \exp(-x))$
- $f'(x) = 1 - f^2(x)$



Squashing Functions

- Sigmoids, step functions, and other functions that force their results to be in a limited range are called “squashing functions”.
- It is generally accepted that biological neural system is based on such functions, since there are physical limits to the response level.