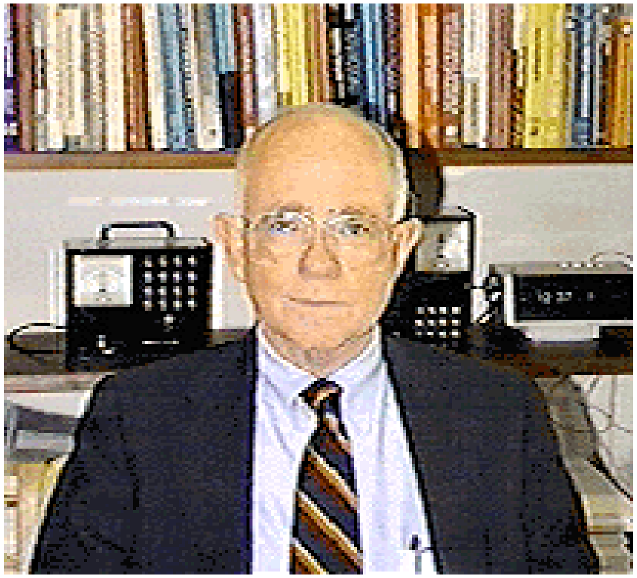
Adalines

Primitive Artificial Neurons

Sweet Adaline

- The Adaline (Adaptive Linear Neuron or Adaptive Linear Element) is a model similar to the Perceptron. There are several variations:
  - One has the threshold function similar to a perceptron.
  - Another uses a **pure linear** function with no threshold.

Adaline Inventors
Bernard Widrow and Marcian (Ted) Hoff

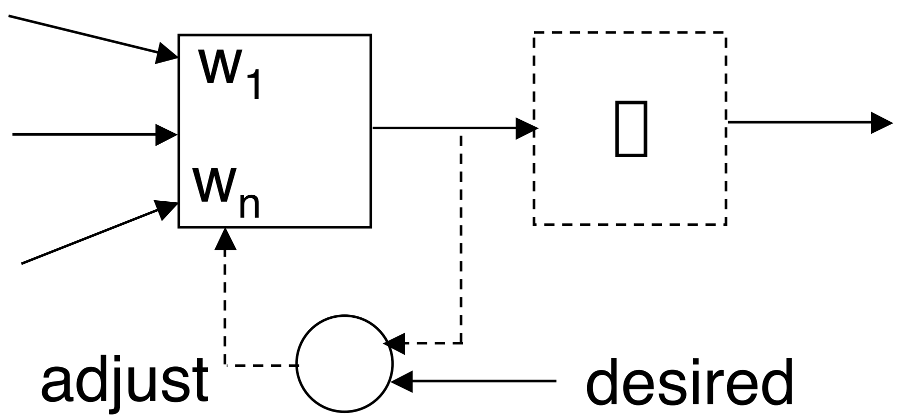


Bernard Widrow,
Professor Emeritus of E.E.,
Stanford University

Marcian Hoff
Co-inventor of Patent 3,821,715
Microprocessor Concept and Architecture

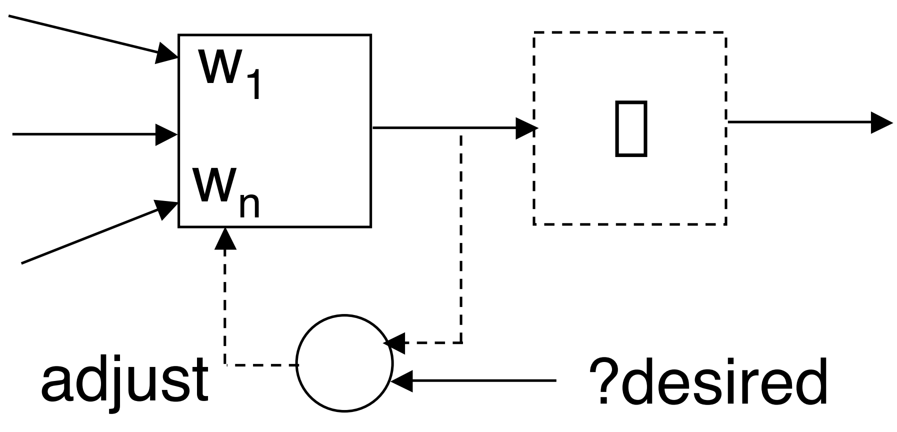
Adaline Training (1)

- With or without the threshold, the Adaline is **trained** based on the output of the linear function rather than the final output.



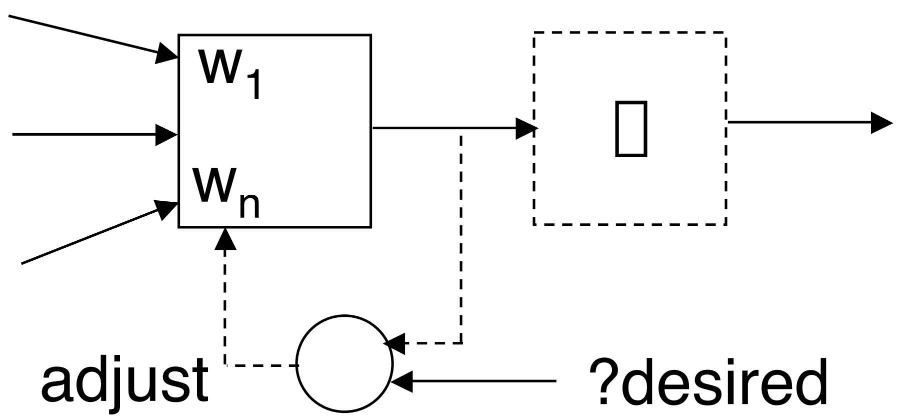
Adaline Training (2)

- The catch here is that we have to state the **desired** value in terms of the output of the linear part, rather than the output after the threshold.
- What is this for a classifier?



Adaline Training (2)

- A reasonable approach is to use a *nominal* value such as -0.5 as desired for a “no” classification and a +0.5 for a “yes” classification.



Adaline Training (3)

- The formula for Adaline weight updating is very similar to the Perceptron:

  Add to the weights □w where

  $$\square w = \square\square\,[\,-1,\ x_1,\ x_2\,,\ \ldots,\ x_n\,]$$

  Adaline learning rule

  only now the ***error* is not limited to 1, -1, 0 as before**; it can have a fractional value, since it is based on the output of the linear part of the device.

Adaline Training (4)

- The weight update formula for the Adaline will be justified eventually.
- One major difference from this vs. the Perceptron is that a learning rate of 1 won't generally be acceptable. It will need to be smaller, say 0.01. There is a theory that tells us how large we can make it.

Adaline Example

- We'll use the same example as before. But now we'll train on the output of the linear portion and target for +1 for a "yes" answer and -1 for a "no" answer.
  - (4, 5) +1
  - (6, 1) +1
  - (4, 1) -1
  - (1, 2) -1
- Try a learning rate of 0.01.

Adaline Training Example

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	1			
	(-1, 6, 1)	1			
	(-1, 4, 1)	-1			
	(-1, 1, 2)	-1			

Adaline Training Example

weights	input	desired	actual	error	new weights
(0, 1, -1)	(-1, 4, 5)	1	-1	2	
	(-1, 6, 1)	1			
	(-1, 4, 1)	-1			
	(-1, 1, 2)	-1			(0.0358, 0.727, -0.943)

.
.
.

We stopped (after 500 epochs) at (0.67, 0.21, 0.17)

weights	input	desired	actual	error
(0.67, 0.21, 0.17)	(-1, 4, 5)	1	1.02	-0.02
	(-1, 6, 1)	1	0.75	0.25
	(-1, 4, 1)	-1	0.33	-1.33
	(-1, 1, 2)	-1	-0.12	-0.88

With a threshold of 0.67, this would correctly separate the samples.

Adaline Convergence (2)

- The Adaline admits a more refined stopping criterion:
  The **Mean-Squared Error (MSE)** is the average of the squares of the error taken over all samples. Squaring makes the measure insensitive to the sign of the error. It also provides certain analytic properties.
- This quantity ideally converges toward a specific minimum (which might never be exactly attained). The algorithm can be set to stop when the MSE reaches a desired value.

Alternate Rule Names

- Because the Adaline rule minimizes MSE, it is sometimes called the "**LMS rule**" [LMS = "least mean square"].

- The term "**Delta rule**" is also sometimes used, although this will be seen to be a rule for a more general class of networks.

Minimizing Error

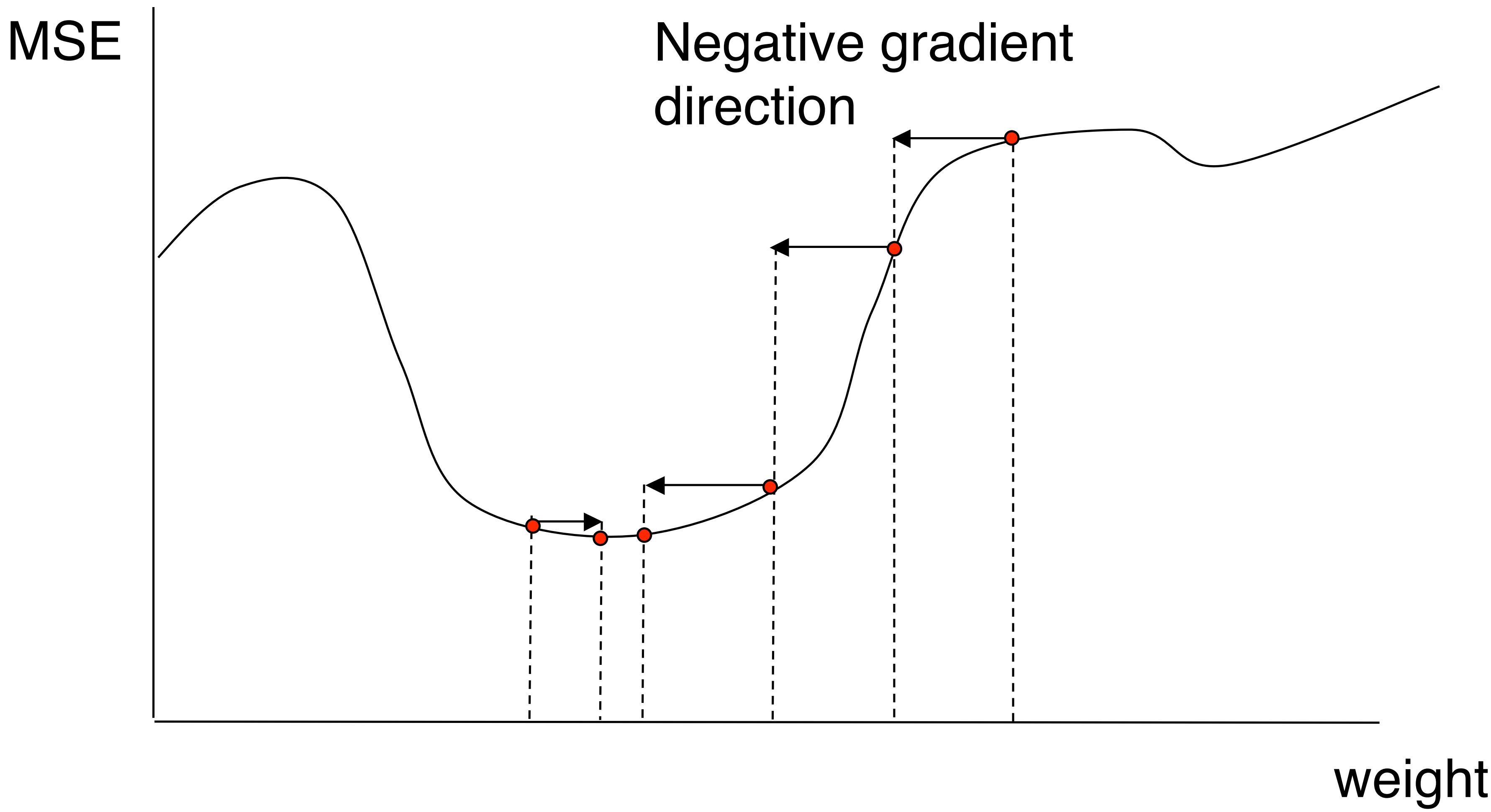
- Consider the task of finding the minimum of a function of one variable.

- One standard method for doing this, if the derivative of the function is known, is Newton's method, which entails constructing a tangent line from a current estimate, then using the intersection of the line with the origin for the next.

Gradient Descent

- Gradient descent is another method for finding the minimum.
- It consists of computing the gradient of the function, then taking a small step in the direction of negative gradient, which hopefully corresponds to decreased function value, then repeating for the new value of the dependent variable.

Gradient Descent

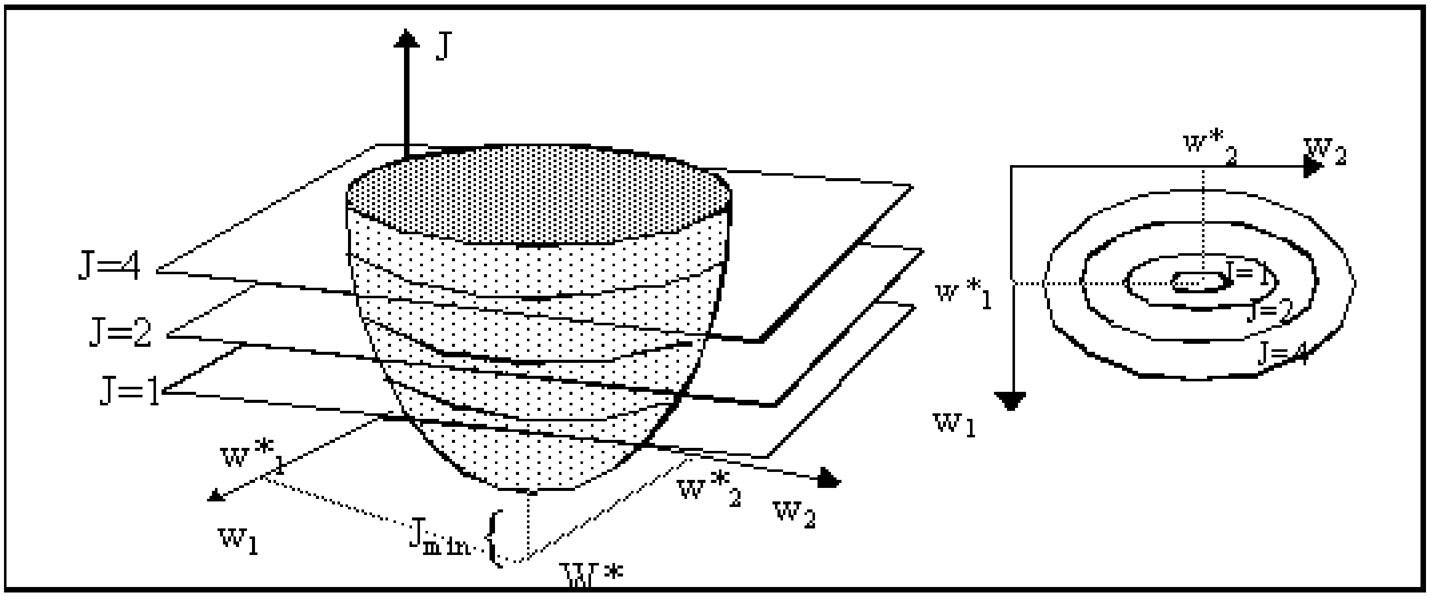


Gradient Descent

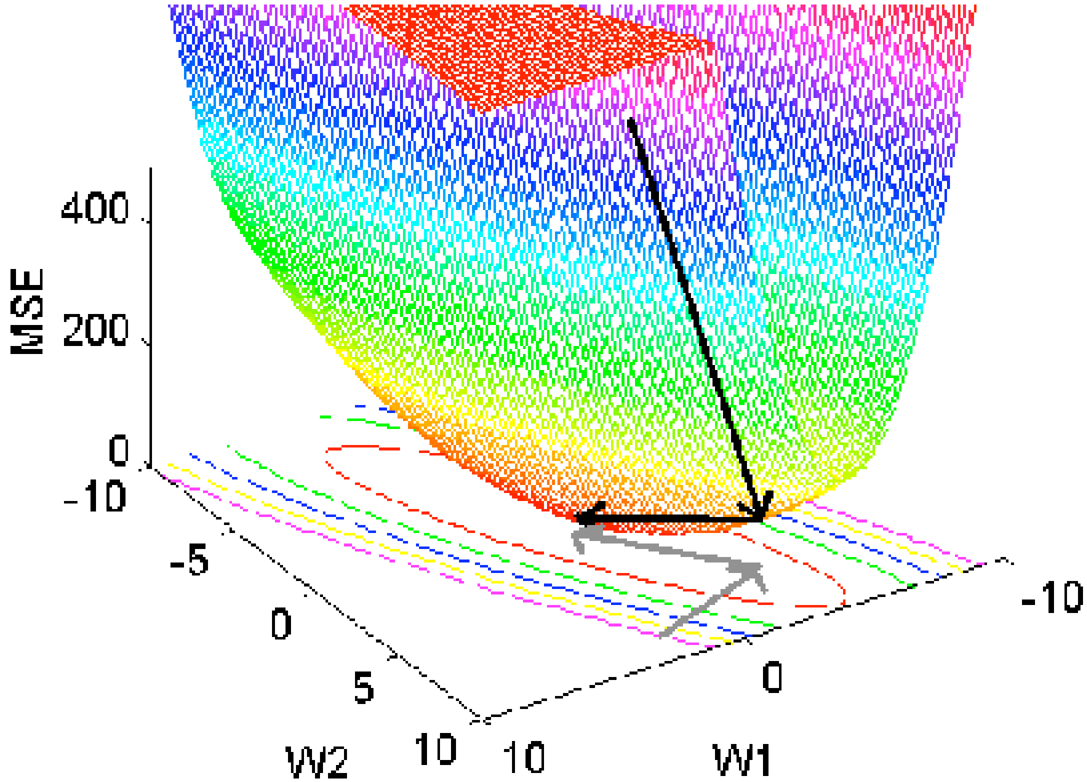
- A single dimension for weights (including bias) is atypical.
- So the previous diagram is mainly to enhance our intuition.
- For the general case, the gradient is a **vector** of gradient components, one for each weight (including bias).

Error vs. 2-D Weight Space

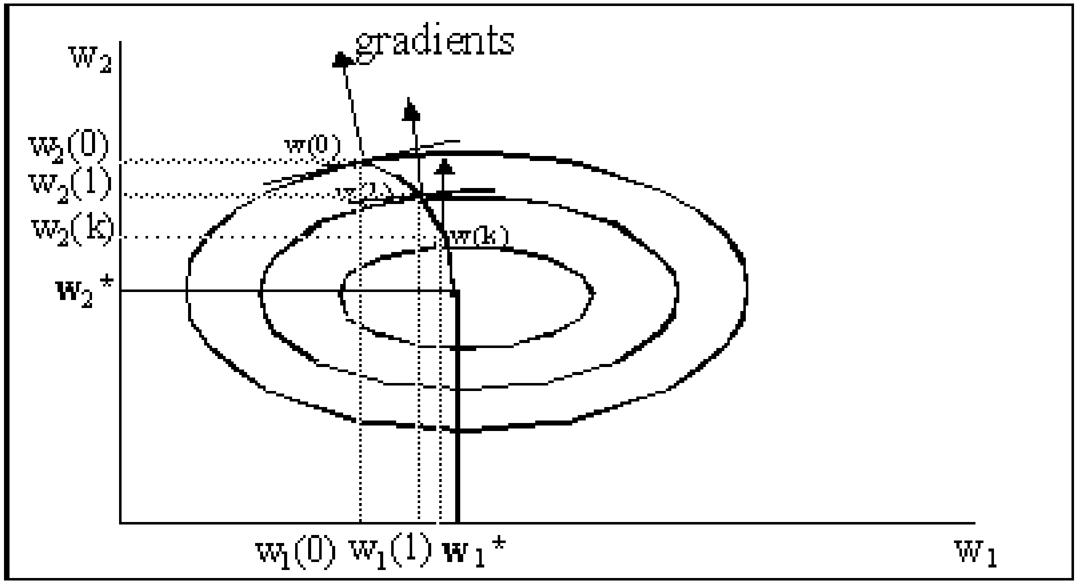
(one weight is threshold or -bias)



2-D Gradient Descent



2-D Gradient Descent



Computing Gradients

Note: squared

- MSE = $J(w) = \Sigma(\text{desired}-\text{actual})^2/n$ where Σ is over n samples.
- desired is a fixed constant for each sample.
- actual = $\Sigma w_j x_j$ (Σ over input lines)
- So $J(w) = \Sigma(\text{desired}- \Sigma w_j x_j)^2/n$

On-Line Approximation to Gradient

- On-line means based on a single sample, rather than batch, which means using all samples
- $J \approx (d - \Sigma w_j x_j)^2$ ← (note: no outer sum)
  (d = desired)
- i^{th} gradient component $= \partial J / \partial w_i$
  $= \partial / \partial w_i (d - \Sigma w_j x_j)^2$
  $= 2 \underbrace{(d - \Sigma w_j x_j)}_{= \text{error}, \delta} \underbrace{\partial / \partial w_i (d - \Sigma w_j x_j)}_{-x_i} = -2\delta x_i$

Computing Gradients

- i^{th} gradient component = $-2\ □\ x_i$
- However we want to move in the *direction* of ***negative*** gradient, tempered by the learning rate □, so:

  Amount to *add* to weight is

  $$\boxed{\Delta w_i = 2\ □\ □\ x_i}$$

  which we recognize as the LMS (Adaline) rule (2 can be folded into □).

Vector Version of the Analysis of Gradient Descent for Adaline

- MSE = $J(w) = E[(d - w^T x)^2]$ (E = expectation or mean averaged over samples)

  $= E[d^2 - 2dw^T x + w^T x\, x^T w]$

  $= E[d^2] - E[2dw^T x] + E[w^T x\, x^T w]$

  $= E[d^2] - 2w^T E[dx] + w^T E[x\, x^T]\, w$

  $= c - 2w^T h + w^T R\, w$, for appropriate const. c, h, R

Vector Analysis of Gradient Descent for Adaline

- $J(w) = c - 2w^T h + w^T R\, w$

  where $c = E[d^2]$, $h = E[d\, x]$, $R = E[x\, x^T]$

- This is a **quadratic form** in w with coefficients derived from the data vectors x.

- R is called the (auto-)**correlation matrix.**

Standard Quadratic Form

- $J(w) = c + w^T b + (1/2) w^T A\, w$

  where $c = E[d^2]$,
  $b = -2E[d\, x]$,
  $A = 2E[x\, x^T]$

- A is called the **Hessian** matrix. It is the matrix of 2nd partial derivatives of the surface.

Analytic Gradient

- $\nabla J(w) = \nabla(c + w^T b + (1/2) w^T A w) =$
  $b + A w$

- It can be shown that if J has a **minimum**, it will be at a point w^* where $\nabla J(w^*) = 0$,
  i.e. $b + A w^* = 0$
  i.e. $\boxed{w^* = A^{-1} b}$

  where $A = 2E[x x^T]$, $b = -2E[dx]$

Stable Points

- In general, $w^* = A^{-1} b$ is a stable point.
- It may correspond to a minimum, maximum, or saddle.

Convergence of Gradient Descent for Adaline

- $\Delta w = 2\ \square\square\ x$
  $w(k+1) = w(k) + 2\ \square\ \square(k)\ x(k)$ (at k^{th} step)

- $E[w(k+1)] = E[w(k)] + 2\ \square\ E[\square(k)\ x(k)]$
  … math ...
  $= (I-2\square R)\ E[w(k)] + 2\square h$

- For convergence, the eigenvalues of the matrix I–2□R must be within the unit circle.

Convergence of Gradient Descent for Adaline

- If λ_i is an eigenvalue of R, convergence requires $|1 - 2\,\eta\,\lambda_i| < 1$.
- which simplifies to $\eta < 1/\lambda_i$ for all eigenvalues λ_i, in particular for the maximum one.

$$\boxed{\eta < 1/\lambda_{max}}$$

Bound on learning rate for convergence of Adaline training by gradient descent

NAS repeatedly uses $2/\lambda_{max}$, possible because it doesn't keep the factor of 2 in the learning rule.

Example
(from NND)

$$\text{Sample 1} \quad \left\{ \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_1 = \begin{bmatrix} -1 \end{bmatrix} \right\} \qquad \text{Sample 2} \quad \left\{ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \end{bmatrix} \right\}$$

$$\mathbf{R} = E[\mathbf{x}\mathbf{x}^T] = \frac{1}{2}\mathbf{x}_1\mathbf{x}_1^T + \frac{1}{2}\mathbf{x}_2\mathbf{x}_2^T$$

$$\mathbf{R} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda_1 = 1.0, \qquad \lambda_2 = 0.0, \qquad \lambda_3 = 2.0$$

$$\alpha < \frac{1}{\lambda_{max}} = \frac{1}{2.0} = 0.5$$

Training, 1st Epoch

Sample 1 $\quad a(0) = \mathbf{W}(0)\mathbf{p}(0) = \mathbf{W}(0)\mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = 0$

$$\square(0) = d(0) - a(0) = d_1 - a(0) = -1 - 0 = -1$$

$$\mathbf{W}(1) = \mathbf{W}(0) + 2\square\square(0)\mathbf{x}^T(0)$$

$$\mathbf{W}(1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + 2(0.2)(-1)\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix}$$

2nd Epoch

Sample 2 $\quad a(1) = \mathbf{W}(1)\mathbf{p}(1) = \mathbf{W}(1)\mathbf{p}_2 = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -0.4$

$$\square(1) = d(1) - a(1) = d_2 - a(1) = 1 - (-0.4) = 1.4$$

$$\mathbf{W}(2) = \begin{bmatrix} 0.4 & -0.4 & 0.4 \end{bmatrix} + 2(0.2)(1.4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix}$$

3rd Epoch

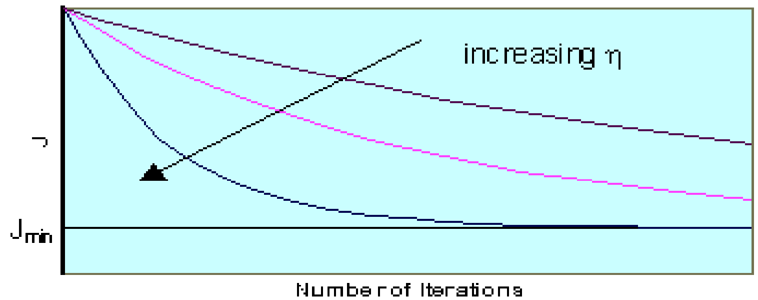
$$a(2) = \mathbf{W}(2)\mathbf{p}(2) = \mathbf{W}(2)\mathbf{p}_1 = \begin{bmatrix} 0.96 & 0.16 & -0.16 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -0.64$$

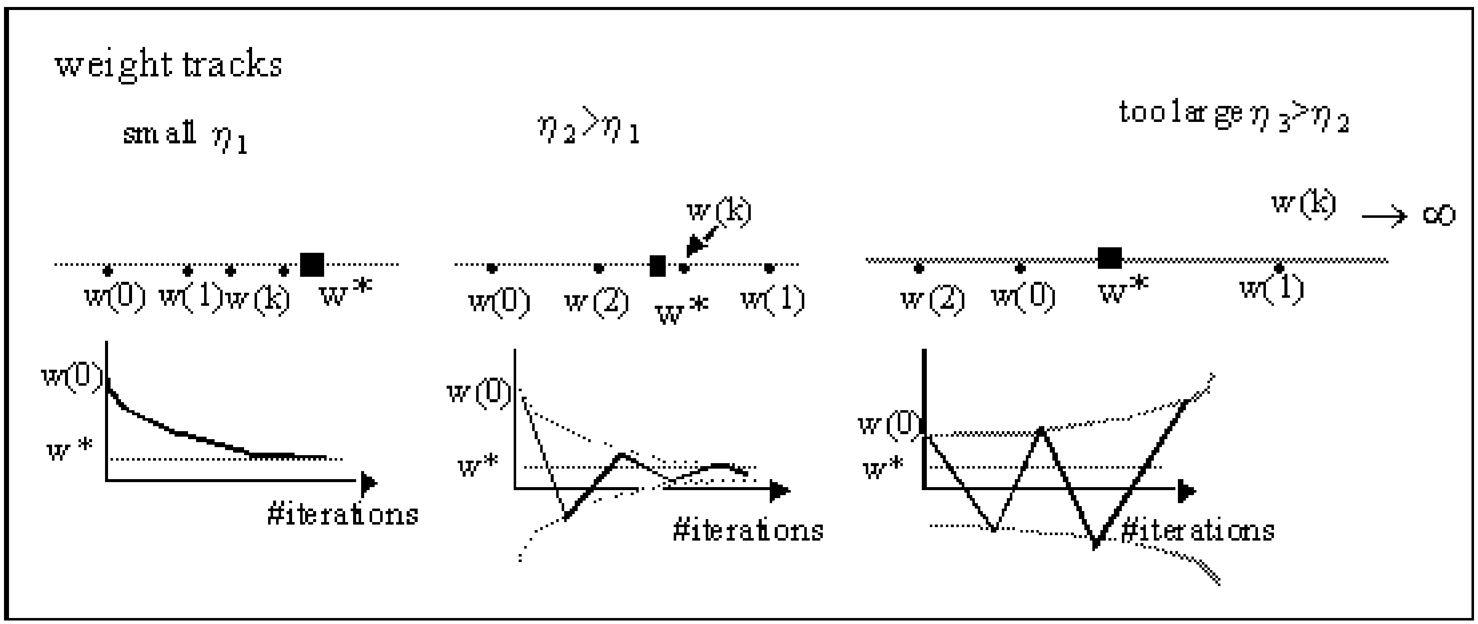
$$e(2) = d(2) - a(2) = d_1 - a(2) = -1 - (-0.64) = -0.36$$

$$\mathbf{W}(3) = \mathbf{W}(2) + 2\alpha e(2)\mathbf{x}^T(2) = \begin{bmatrix} 1.1040 & 0.0160 & -0.0160 \end{bmatrix}$$

$$\mathbf{W}(\ ) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

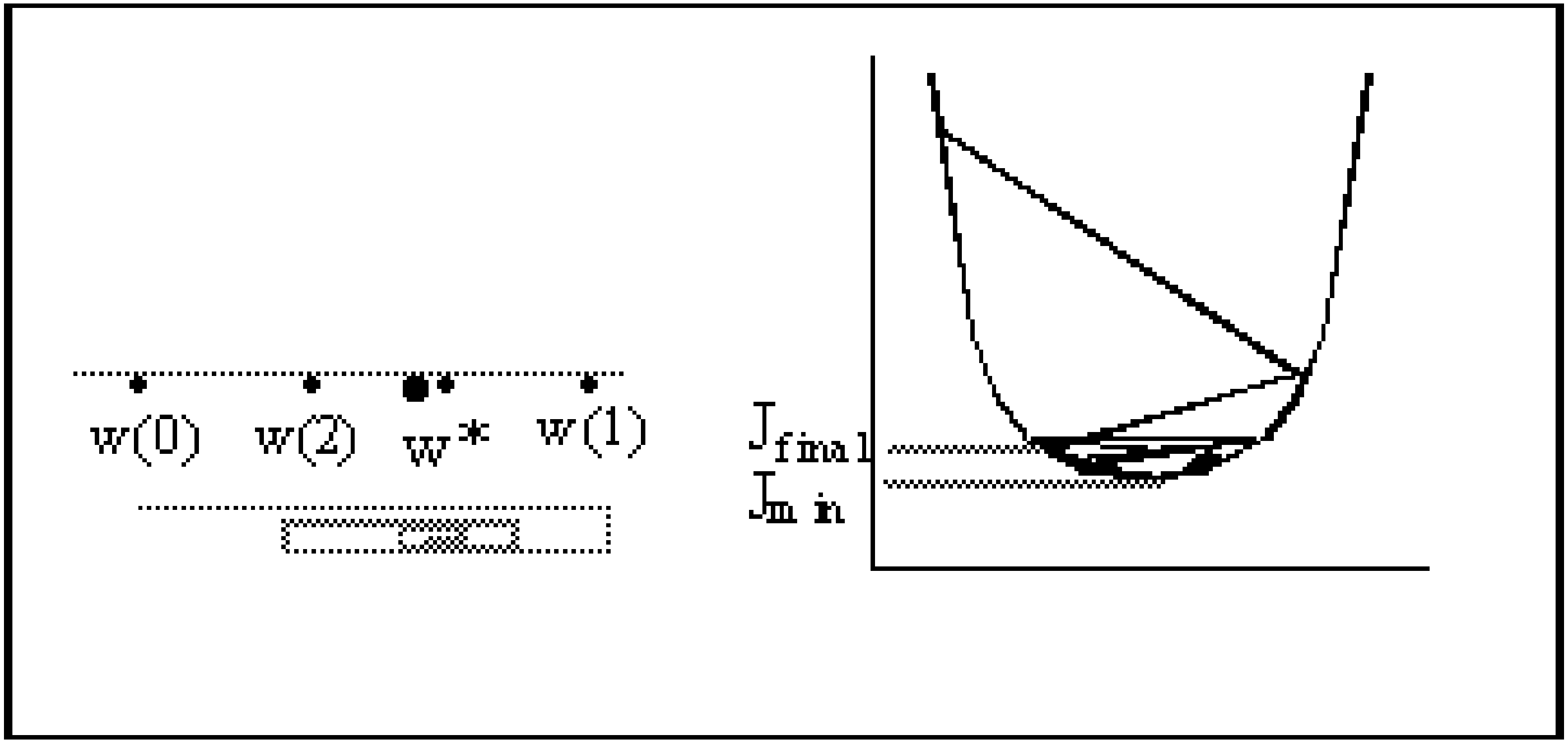
"Learning Curve"





“Rattling”

If learning rate too high, minimum solution won’t be achieved, even if there is no divergence.

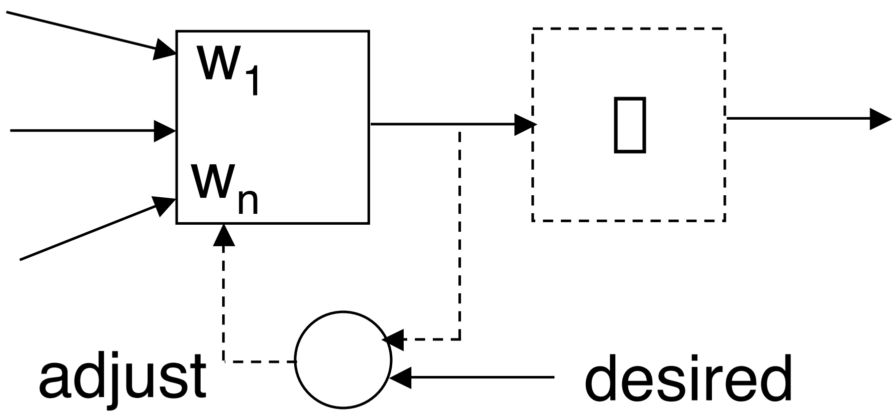


Solutions to Rattling

- Rule of thumb: Use a learning rate 0.1 times the theoretical maximum, or
- Gradually decrease the learning rate, e.g. according to a pre-set schedule, or
- Adaptively set the learning rate:
  - If the MSE is taking big steps, make it smaller.
  - If the MSE is taking small steps, make it larger.

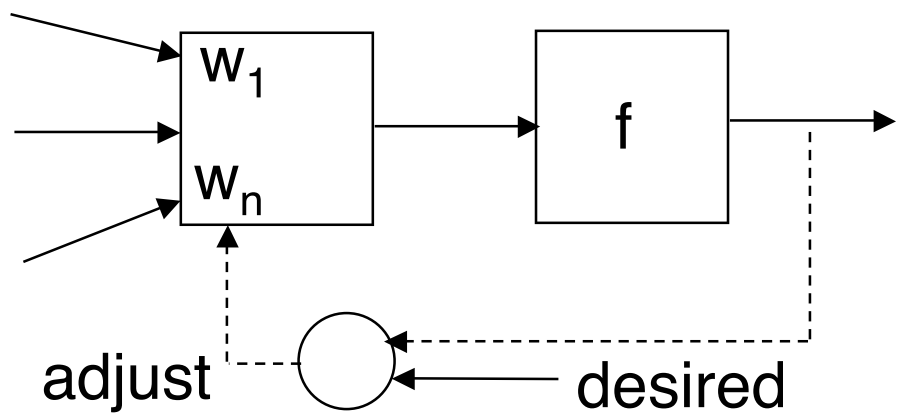
Generalizing the Adaline

- We have already discussed the problem of a threshold output in this picture



Generalizing the Adaline

- Suppose that we replace the threshold stage with a general function f and revert to expressing desired in terms of its output:



Generalizing the Adaline

- Consider again the derivation of the LMS rule:
  $J(w) = \square(\text{desired} - f(\square w_j x_j))^2/n$
- $J \approx (d - f(\square w_j x_j))^2$ (d = desired)
- i^{th} gradient component $= \partial J/\partial w_i$
  $= \partial/\partial w_i (d - f(\square w_j x_j))^2$
  $= 2\,(d - f(\square w_j x_j))\ \partial/\partial w_i\ (d - f(\square w_j x_j))$
  $= -2\ \square\ \partial/\partial w_i\ f(\square w_j x_j)$
  $= -2\ \square\ f'(\square w_j x_j)\ \partial/\partial w_i\ \square w_j x_j = -2\ \square\ x_i f'(\square w_j x_j)$

Generalized LMS Rule
(or Delta Rule)

- $\Delta w = 2 \; \square \; \square \; f'(\square w_j x_j) \; x_i$
  assuming that f has a derivative f'.

- $\square w_j x_j$ is often called the **"net" value** or **"activation" value**, and f the **activation function**.

- For the special case of f being the identity function, this reduces to the LMS rule we had before.

Generalized LMS Rule
(or Delta Rule)

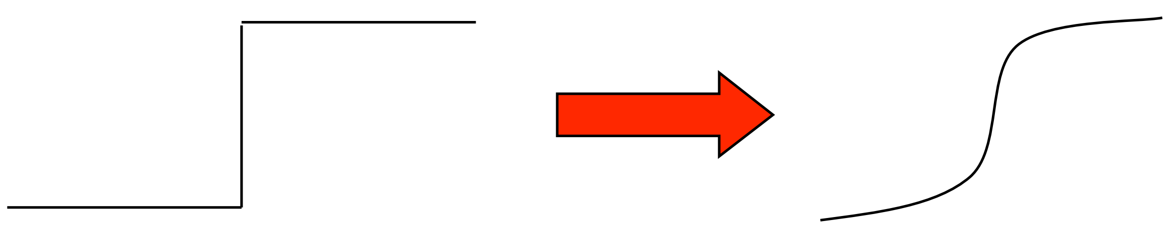
- Why worry about this generalization?

  $\Delta w = 2 \, \square \, \square \, f'(\square w_j x_j) \, x_i$

- It will have a number important uses.

Use #1

- In the Adaline with threshold, we can't very well treat the model analytically, due to the fact that we have a non-continuous function at the output.
- But we can *approximate* the non-continuous function with a continuous one:



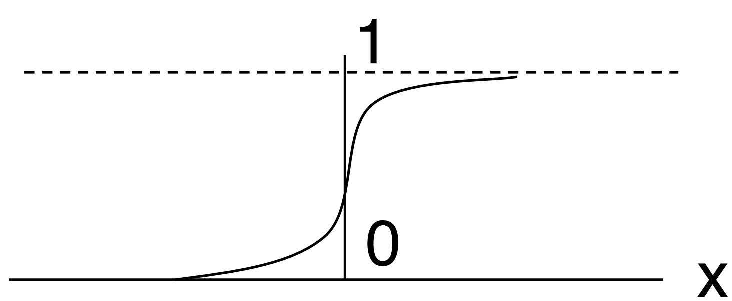
Sigmoids

- The “S” shape on the right of the previous slide is called a sigmoid curve.
- This is a generic term and there are several different analytic functions that behave this way.

Logisitic Sigmoid

- Logisitic function (“logsig”–Matlab):
  f(x) = 1/(1+exp(-ax))

- f’(x) = f(x)(1-f(x))



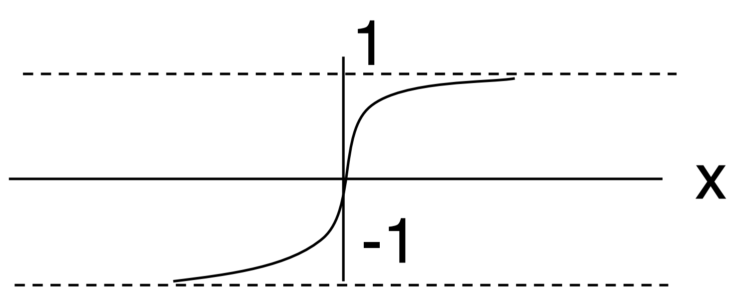
Hyperbolic Sigmoid

- Hyperbolic tangent function (“tansig”):

  $f(x) = \tanh(x)$

  $= (\exp(x)-\exp(-x))/(\exp(x) + \exp(-x))$

- $f'(x) = 1-f^2(x)$



Squashing Functions

- Sigmoids, step functions, and other functions that force their results to be in a limited range are called “squashing functions”.

- It is generally accepted that biological neural system is based on such functions, since there are physical limits to the response level.