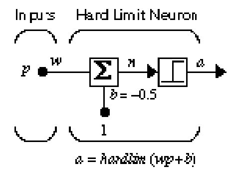
Associative Learning

(Unsupervised Hebbian and others)

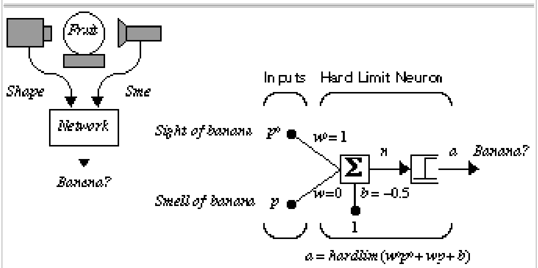
Simple Associative Network



$$a = hardlim(wp + b) = hardlim(wp - 0.5)$$

$$p = \begin{cases} 1, & \text{stimulus} \\ 0, & \text{no stimulus} \end{cases} \qquad a = \begin{cases} 1, & \text{response} \\ 0, & \text{no response} \end{cases}$$

Banana Associator



Unconditioned Stimulus

$$p^0 = \begin{cases} 1, & \text{shape detected} \\ 0, & \text{shape not detected} \end{cases}$$

Conditioned Stimulus

$$p = \begin{cases} 1, & \text{smell detected} \\ 0, & \text{smell not detected} \end{cases}$$

Unsupervised Hebb Rule

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q)$$

Vector Form:

$$\boxed{\mathbf{W}(q) = \mathbf{W}(q-1) + \alpha \mathbf{a}(q)\mathbf{p}^T(q)}$$

Training Sequence:

$$\mathbf{p}(1), \mathbf{p}(2), \ldots, \mathbf{p}(Q)$$

Banana Recognition Example

Initial Weights:

$$w^0 = 1, w(0) = 0$$

Training Sequence:

$$\{p^0(1) = 0, p(1) = 1\}, \{p^0(2) = 1, p(2) = 1\}, \ldots$$

$$\alpha = 1$$

$$w(q) = w(q-1) + a(q)p(q)$$

First Iteration (sight fails):

$$a(1) = hardlim(w^0 p^0(1) + w(0)p(1) - 0.5)$$
$$= hardlim(1 \cdot 0 + 0 \cdot 1 - 0.5) = 0 \quad \text{(no response)}$$

$$w(1) = w(0) + a(1)p(1) = 0 + 0 \cdot 1 = 0$$

Example

Second Iteration (sight works):

$$a(2) = hardlim(w^0 p^0(2) + w(1)p(2) - 0.5)$$
$$= hardlim(1 \cdot 1 + 0 \cdot 1 - 0.5) = 1 \quad \text{(banana)}$$

$$w(2) = w(1) + a(2)p(2) = 0 + 1 \cdot 1 = 1$$

Third Iteration (sight fails):

$$a(3) = hardlim(w^0 p^0(3) + w(2)p(3) - 0.5)$$
$$= hardlim(1 \cdot 0 + 1 \cdot 1 - 0.5) = 1 \quad \text{(banana)}$$

$$w(3) = w(2) + a(3)p(3) = 1 + 1 \cdot 1 = 2$$

Banana will now be detected if either sensor works.

Problems with Hebb Rule

- Weights can become arbitrarily large
- There is no mechanism for weights to decrease

Hebb Rule with Decay

$$\mathbf{W}(q) = \mathbf{W}(q-1) + \alpha\mathbf{a}(q)\mathbf{p}^T(q) - \gamma\mathbf{W}(q-1)$$

$$\boxed{\mathbf{W}(q) = (1-\gamma)\mathbf{W}(q-1) + \alpha\mathbf{a}(q)\mathbf{p}^T(q)}$$

This keeps the weight matrix from growing without bound, which can be demonstrated by setting both a_i and p_j to 1:

$$w_{ij}^{max} = (1-\gamma)w_{ij}^{max} + \alpha a_i p_j$$
$$w_{ij}^{max} = (1-\gamma)w_{ij}^{max} + \alpha$$
$$w_{ij}^{max} = \frac{\alpha}{\gamma}$$

Example: Banana Associator

$\alpha = 1$ $\gamma = 0.1$

First Iteration (sight fails):

$$a(1) = hardlim(w^0 p^0(1) + w(0)p(1) - 0.5)$$
$$= hardlim(1\cdot 0 + 0\cdot 1 - 0.5) = 0 \quad \text{(no response)}$$

$$w(1) = w(0) + a(1)p(1) - 0.1w(0) = 0 + 0\cdot 1 - 0.1(0) = 0$$

Second Iteration (sight works):

$$a(2) = hardlim(w^0 p^0(2) + w(1)p(2) - 0.5)$$
$$= hardlim(1\cdot 1 + 0\cdot 1 - 0.5) = 1 \quad \text{(banana)}$$

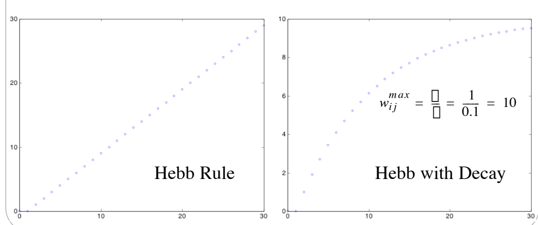
$$w(2) = w(1) + a(2)p(2) - 0.1w(1) = 0 + 1\cdot 1 - 0.1(0) = 1$$

Example

Third Iteration (sight fails):

$$a(3) = hardlim(w^0 p^0(3) + w(2)p(3) - 0.5)$$
$$= hardlim(1\cdot 0 + 1\cdot 1 - 0.5) = 1 \quad \text{(banana)}$$

$$w(3) = w(2) + a(3)p(3) - 0.1w(3) = 1 + 1\cdot 1 - 0.1(1) = 1.9$$



Problem of Hebb with Decay

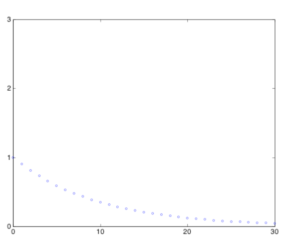
- Associations will decay away if stimuli are not occasionally presented.

If $a_i = 0$, then

$$w_{ij}(q) = (1-\gamma)w_{ij}(q-1)$$

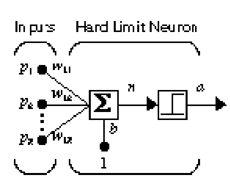
If $\gamma = 0$, this becomes

$$w_{ij}(q) = (0.9)w_{ij}(q-1)$$



Therefore the weight decays by 10% at each iteration where there is no stimulus.

Instar (Recognition Network)



Instar Operation

$$a = hardlim(\mathbf{W}\mathbf{p} + b) = hardlim({}_1\mathbf{w}^T\mathbf{p} + b)$$

The instar will be active when

$${}_1\mathbf{w}^T\mathbf{p} \geq -b$$

or

$${}_1\mathbf{w}^T\mathbf{p} = \|{}_1\mathbf{w}\|\|\mathbf{p}\|\cos\theta \geq -b$$

For normalized vectors, the largest inner product occurs when the angle between the weight vector and the input vector is zero -- the input vector is equal to the weight vector.

> The rows of a weight matrix represent patterns to be recognized.

Vector Recognition

If we set

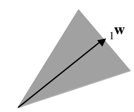
$$b = -\|{}_1\mathbf{w}\|\|\mathbf{p}\|$$

the instar will only be active when $\theta = 0$.

If we set

$$b > -\|{}_1\mathbf{w}\|\|\mathbf{p}\|$$

the instar will be active for a range of angles.



As b is increased, the more patterns there will be (over a wider range of θ) which will activate the instar.

Instar Rule

Hebb with Decay

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q)$$

Modify so that learning and forgetting will only occur when the neuron is active - Instar Rule:

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q) p_j(q) - \gamma a_i(q) w_{ij}(q-1)$$

or

$$w_{ij}(q) = w_{ij}(q-1) + \alpha a_i(q)(p_j(q) - w_{ij}(q-1))$$

Vector Form:

> $${}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1) + \alpha a_i(q)(\mathbf{p}(q) - {}_i\mathbf{w}(q-1))$$

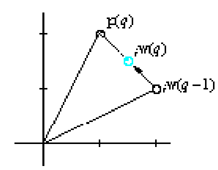
Graphical Representation

For the case where the instar is active ($a_i = 1$):

$${}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1) + \alpha(\mathbf{p}(q) - {}_i\mathbf{w}(q-1))$$

or

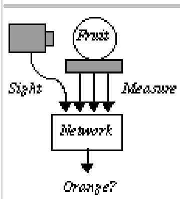
$${}_i\mathbf{w}(q) = (1-\alpha)\,{}_i\mathbf{w}(q-1) + \alpha\mathbf{p}(q)$$



For the case where the instar is inactive ($a_i = 0$):

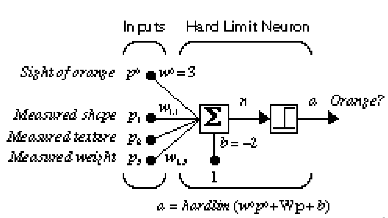
$${}_i\mathbf{w}(q) = {}_i\mathbf{w}(q-1)$$

Example



$$p^0 = \begin{cases} 1, & \text{orange detected visually} \\ 0, & \text{orange not detected} \end{cases}$$

$$\mathbf{p} = \begin{bmatrix} shape \\ texture \\ weight \end{bmatrix}$$



Training

$$\mathbf{W}(0) = {}_1\mathbf{w}^T(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ p^0(1) = 0, \mathbf{p}(1) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\} \left\{ p^0(2) = 1, \mathbf{p}(2) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\} \ldots$$

First Iteration (α=1):

$$a(1) = hardlim(w^0 p^0(1) + \mathbf{W}\mathbf{p}(1) - 2)$$

$$a(1) = hardlim\left(3 \cdot 0 + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - 2 \right) = 0 \quad \text{(no response)}$$

$${}_1\mathbf{w}(1) = {}_1\mathbf{w}(0) + a(1)(\mathbf{p}(1) - {}_1\mathbf{w}(0)) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Further Training

$$a(2) = hardlim(w^0 p^0(2) + \mathbf{W}\mathbf{p}(2) - 2) = hardlim\left(3 \cdot 1 + \begin{bmatrix}0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\-1\end{bmatrix} - 2\right) = 1 \quad \text{(orange)}$$

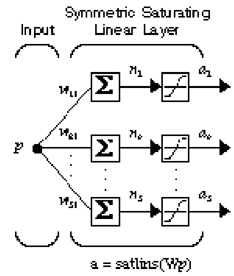
$${}_1\mathbf{w}(2) = {}_1\mathbf{w}(1) + a(2)(\mathbf{p}(2) - {}_1\mathbf{w}(1)) = \begin{bmatrix}0\\0\\0\end{bmatrix} + 1\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$a(3) = hardlim(w^0 p^0(3) + \mathbf{W}\mathbf{p}(3) - 2) = hardlim\left(3 \cdot 0 + \begin{bmatrix}1 & -1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\\-1\end{bmatrix} - 2\right) = 1 \quad \text{(orange)}$$

$${}_1\mathbf{w}(3) = {}_1\mathbf{w}(2) + a(3)(\mathbf{p}(3) - {}_1\mathbf{w}(2)) = \begin{bmatrix}1\\-1\\-1\end{bmatrix} + 1\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix} - \begin{bmatrix}1\\-1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

Orange will now be detected if either set of sensors works.

Outstar (Recall Network)



Outstar Operation

Suppose we want the outstar to recall a certain pattern $\mathbf{a}^*$ whenever the input $p = 1$ is presented to the network. Let

$$\mathbf{W} = \mathbf{a}^*$$

Then, when $p = 1$

$$\mathbf{a} = \mathbf{satlins}(\mathbf{W}p) = \mathbf{satlins}(\mathbf{a}^* \cdot 1) = \mathbf{a}^*$$

and the pattern is correctly recalled.

The columns of a weight matrix represent patterns to be recalled.

Outstar Rule

For the instar rule we made the weight decay term of the Hebb rule proportional to the <u>output</u> of the network. For the outstar rule we make the weight decay term proportional to the <u>input</u> of the network.

$$w_{ij}(q) = w_{ij}(q-1) + \square a_i(q) p_j(q) - \square p_j(q) w_{ij}(q-1)$$

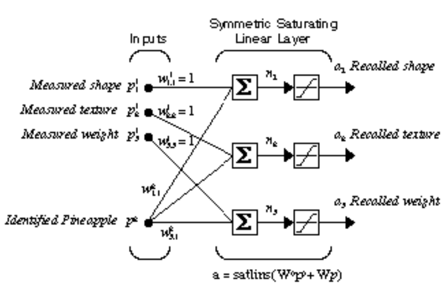
If we make the decay rate □ equal to the learning rate □,

$$w_{ij}(q) = w_{ij}(q-1) + \square(a_i(q) - w_{ij}(q-1)) p_j(q)$$

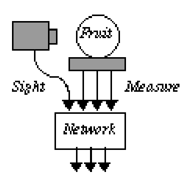
Vector Form:

$$\mathbf{w}_j(q) = \mathbf{w}_j(q-1) + \square(\mathbf{a}(q) - \mathbf{w}_j(q-1)) p_j(q)$$

Example - Pineapple Recall



Definitions



$$\mathbf{a} = \mathbf{satlins}(\mathbf{W}^0\mathbf{p}^0 + \mathbf{W}p)$$

$$\mathbf{W}^0 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\mathbf{p}^0 = \begin{bmatrix}shape\\texture\\weight\end{bmatrix} \qquad \mathbf{p}^{pineapple} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$p = \begin{cases}1, & \text{if a pineapple can be seen}\\0, & \text{otherwise}\end{cases}$$

Iteration 1

$$\left\{\mathbf{p}^0(1) = \begin{bmatrix}0\\0\\0\end{bmatrix}, p(1) = 1\right\} \quad \left\{\mathbf{p}^0(2) = \begin{bmatrix}-1\\-1\\1\end{bmatrix}, p(2) = 1\right\}$$

$$\gamma = 1$$

$$\mathbf{a}(1) = \mathbf{satlins}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}1\right) = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{(no response)}$$

$$\mathbf{w}_1(1) = \mathbf{w}_1(0) + (\mathbf{a}(1) - \mathbf{w}_1(0))\,p(1) = \begin{bmatrix}0\\0\\0\end{bmatrix} + \left(\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right)1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

25

Convergence

$$\mathbf{a}(2) = \mathbf{satlins}\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}1\right) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \quad \text{(measurements given)}$$

$$\mathbf{w}_1(2) = \mathbf{w}_1(1) + (\mathbf{a}(2) - \mathbf{w}_1(1))\,p(2) = \begin{bmatrix}0\\0\\0\end{bmatrix} + \left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right)1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\mathbf{a}(3) = \mathbf{satlins}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\-1\\1\end{bmatrix}1\right) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \quad \text{(measurements recalled)}$$

$$\mathbf{w}_1(3) = \mathbf{w}_1(2) + (\mathbf{a}(2) - \mathbf{w}_1(2))\,p(2) = \begin{bmatrix}-1\\-1\\1\end{bmatrix} + \left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} - \begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

26