Using Energy-Minimizing Networks to Solve Constraint-Satisfaction Problems

Constraint-Satisfaction Problems

- **Minimum-energy seeking networks** (e.g. using annealing) can be used to find solutions to constraint-satisfaction problems (problems of finding optima, subject to constraints that can't be violated, rather than as computing a function).

- A number of other such problems have been studied in this context.

Traveling Salesperson Problem

- The problem is: given a set of n nodes ("cities") with a specified minimum cost between each pair of nodes, find a permutation ("tour") of the nodes that minimizes the summed costs between the nodes in the permutation sequence.

- The costs are symmetric, and the general problem does not require that there be any Euclidean relationship among the nodes.

Finding Solutions to the TSP using a Hopfield Net

- Global minimum is not necessarily found, although this might be doable with a Boltzmann style algorithm instead.
- The difficulty is encoding the instance of the TSP as a net:

minimal cost □ minimal energy

TSP Formulation

- Represent a given problem as a matrix:
  - Cities correspond to rows.
  - Positions on the tour correspond to columns.
  - Example:

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & 0 & 0 & 1 & 0 & 0 \\
B & 1 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 1 \\
D & 0 & 1 & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 1 & 0
\end{array}
$$

means B occurs first on the tour, D occurs second, A third, E fourth, C fifth.

TSP Formulation

- Assume {0, 1} values rather than {-1, 1}.
- The neurons correspond to entries in the matrix (n^2 neurons for n cities).
- Neurons in a row have inhibitory connections from other neurons in same row:
  - If one neuron is on, then others tend to be off, especially in minimum energy state.
- Similarly for neurons in the same column

TSP Formulation

- Need to favor tours that include *all* n cities, as opposed to just a subset of them.
- Need to represent costs between cities as neural weights:
  - Want to inhibit selection of adjacent cities in proportion to the cost between those cities.
  - Let X and Y be rows (cities) and i and j be columns (positions).

TSP Formulation

- Using the expression for energy in a Hopfield net $-\frac{1}{2}\sum\sum w_{ij}\, y_i y_j$, the corresponding energy is computed to have the form

$$A \sum_X \sum_i \sum_{j\neq i} y_{Xi} y_{Xj}$$

$$+ B \sum_i \sum_X \sum_{Y\neq X} y_{Xi} y_{Yi}$$

$$+ C \left(\sum_X \sum_i y_{Xi} - n\right)^2$$

$$+ D \sum_i \sum_X \sum_{Y\neq X} c_{XY} y_{Xi} (y_{Y,i+1} + y_{Y,i-1})$$

- At energy minimum, only the last term, which represents the tour cost, is non-zero.
- We'll explain these terms one at a time.

$A \sum_X \sum_i \sum_{j \neq i} y_{Xi} y_{Xj}$

- The outer summation is over all cities X. The inner summations are over all pairs of distinct positions.
- There is a contribution of +1 if the same city occurs in more than one position in the tour.
- Therefore this term should ideally be 0.

$B \sum_i \sum_X \sum_{Y \neq X} y_{Xi} y_{Yi}$

- The outer summation is over all positions in the tour. The inner summations are over all pairs of distinct cities in position i.
- There is a contribution of +1 if the same city occurs in more than one position in the tour.
- Therefore this term should ideally be 0.

$C\ (\Sigma_X \Sigma_i\ y_{Xi} - n)^2$

- This term tries guarantees that all cities get used. If the summation is n, the term is 0. If it is less than n, the term will be positive.
- This term should ideally be 0.

$D \sum_i \sum_X \sum_{Y \neq X} c_{XY} y_{Xi} (y_{Y,i+1} + y_{Y,i-1})$

- This term represents the cost of the tour. The outer sum is over all positions in the tour, the inner sums over all distinct pairs.
- c_{XY} represents the cost of going from X to Y. This term gets added provided X is at the i^{th} position in the tour, represented by $y_{Xi} = 1$, **and** Y is either at the $(i+1)^{th}$ or $(i-1)^{th}$ position (it can't be at both, by the other constraints). (i+1) and (i-1) are computed mod n.

Weight Derivation

- In order to get the energy function to come out as specified, choose the weight from Xi to Yj as

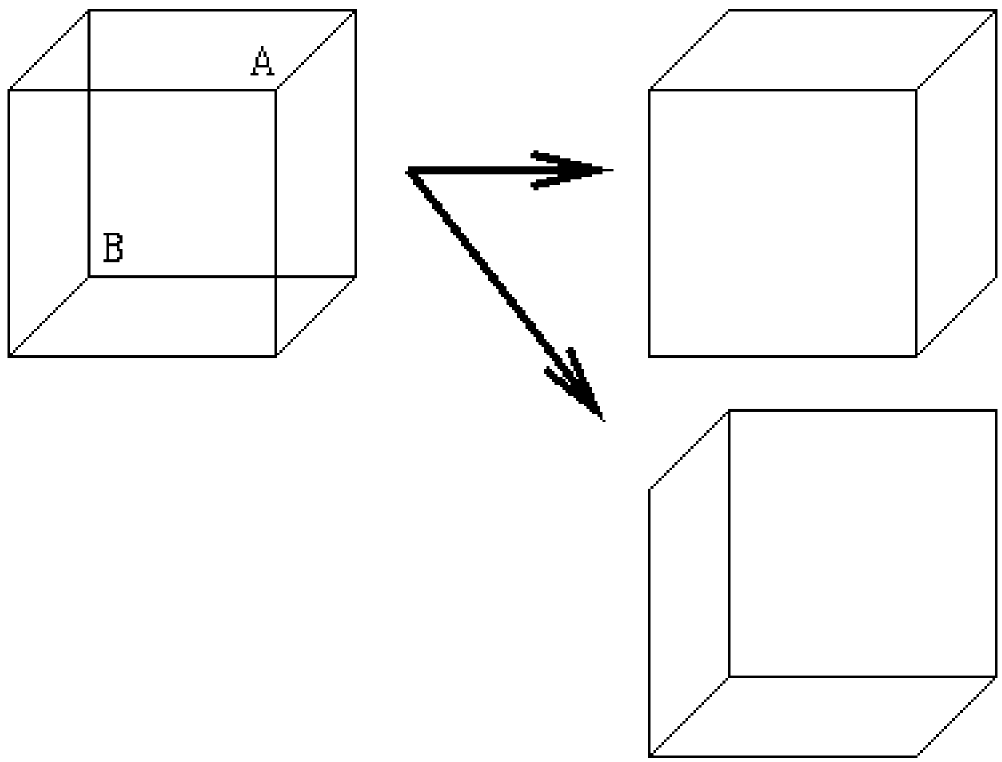
$$w_{XiYj} = -A\delta_{XY}(1-\delta_{ij}) - B\delta_{ij}(1-\delta_{XY}) - C - Dc_{XY}(\delta_{j,i+1} + \delta_{j,i-1})$$

  for appropriate constants A, B, C, D.

- $\delta_{j,i}$ is the Kronecker delta (1 if i = j, 0 otherwise)

Necker Cube “Boltzmann” Demo

(http://www.cs.cf.ac.uk/Dave/JAVA/boltzman/Necker.html)



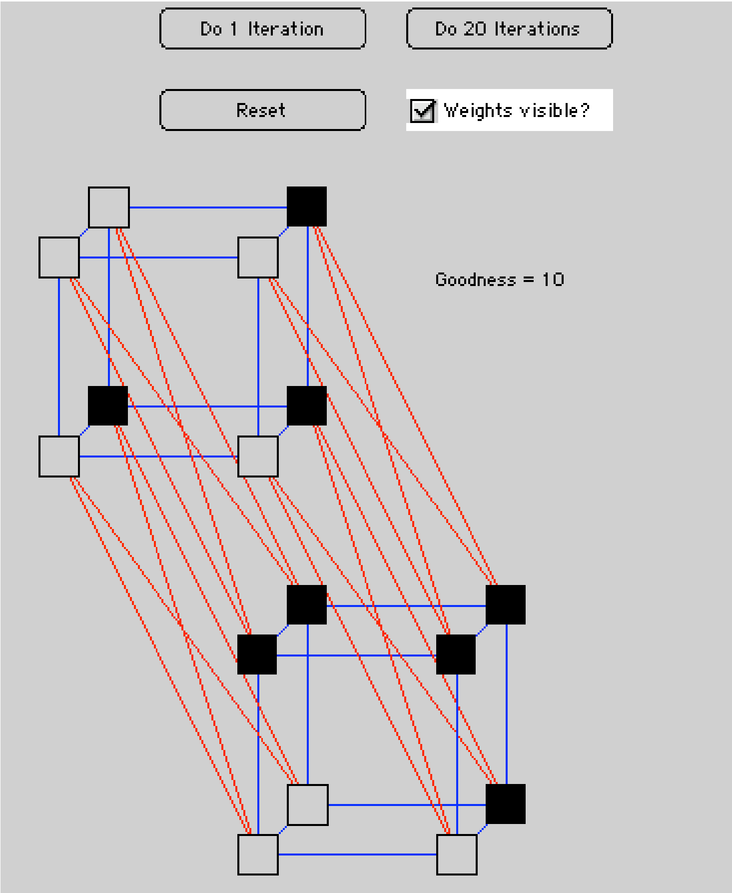
Constraints Represented

- Each vertex can have only one (of two possible) interpretation. Therefore there are negative weights connecting units representing different interpretations of the same vertex.

- The same interpretation cannot be given to more than one vertex - so units representing the same interpretation are connected with negative weights.

- Units from the same interpretation should be on together, so locally consistent units are connected with positive weights.

Necker Cube Demo



Neurons representing whether vertex is shown in **front** or **back**

Blue weights are positive, red are negative.