Natural Deduction Examples for Number Theory

Let □ denote the set of axioms for number theory, given at the end.

1. □ |□ (□y) (0 + y) = y

1.	0 + 0 = 0	□x e A3
2.	n	
3.	(0 + n) = n	Assume
4.	S(0 + n) = S(0 + n)	=i
5.	S(0 + n) = S(n)	=e 3, 4
1.	(□y) 0 + S(y) = S(0 + y)	□x e A4
7.	0 + S(n) = S(0 + n)	□y e 6
8.	0 + S(n) = S(n)	=e 5, 7
9.	((0 + n) = n) □ (0 + S(n) = S(n))	□ i 3-8
10.	(□n) (0 + n = n) □ (0 + S(n) = S(n))	□n i 2-9
11.	(0 + 0 = 0) □ (□n) ((0 + n) = n) □ (0 + S(n) = S(n)))	□i 1, 10
12.	((0 + 0 = 0) □ (□n) ((0 + n) = n) □ (0 + S(n) = S(n))) □ (□n) (0 + n = n)	A7
13.	(□n) (0 + n = n)	□ e 11, 12
14.	y_1	
15.	$(0 + y_1 = y_1)$	□n e 13
16.	(□y) (0 + y = y)	□y i 14-15

This was my original "proof", but it doesn't really match the template for A7, and is not correct.

1.	(□x) (x + 0) = x	A3
2.	0 + 0 = 0	□x e 1
3.	(□y) (0 + y) = y	Assume
4.	y_0	
5.	$0 + S(y_0) = S(y_0)$	□y e 3
6.	(□y) (0 + S(y)) = S(y)	□y i 4-5
7.	((□y) (0 + y) = y) □ ((□y) (0 + S(y)) = S(y))	□ i 3-6
8.	(0 + 0 = 0) □ ((□y) (0 + y) = y) □ ((□y) (0 + S(y)) = S(y))	□i 2, 8
9.	((0 + 0 = 0) □ (□y) (0 + y = y) □ (0 + S(y) = S(y))) □ (□y) (0 + y = y)	A7
10.	(□y) (0 + y = y)	□ e 8, 9

2. □ |□ (□y) (□x) (S(x) + y) = S(x + y)

1.	x_0	
2.	(□x) (x + 0) = x	A3
3.	S(x_0) + 0 = S(x_0)	□x e 2
4.	x_0 + 0 = x_0	□x e 2
5.	S(x_0) + 0 = S(x_0 + 0)	=e 4, 3
6.	(□x) (S(x) + 0) = S(x + 0)	□x i 1-5
7.	n	
8.	(□x) (S(x) + n) = S(x + n)	Assume
9.	x_1	
10.	(S(x_1) + n) = S(x_1 + n)	□x e 8
11.	(□x) (□y) (x + S(y)) = S(x+y)	A4
12.	(□y) (S(x_1) + S(y)) = S(S(x_1)+y)	□x e 11
13.	(S(x_1) + S(n)) = S(S(x_1)+n)	□y e 12
14.	(S(x_1) + S(n)) = S(S(x_1 + n))	=e 10, 13
15.	(□y) (x_1 + S(y)) = S(x_1+y)	□x e 11
16.	x_1 + S(n) = S(x_1 + n)	□y e 15
17.	(S(x_1) + S(n)) = S(x_1 + S(n))	=e 14, 16
18.	(□x) (S(x) + S(n)) = S(x + S(n))	□x i 9-17
19.	((□x) (S(x) + n) = S(x + n)) □ (□x) (S(x) + S(n)) = S(x + S(n))	□ i 8-18
20.	(□n) (((□x) (S(x) + n) = S(x + n)) □ (□x) (S(x) + S(n)) = S(x + S(n)))	□n i 7-19
21.	((□x) (S(x) + 0) = S(x + 0)) □ (□n) (((□x) (S(x) + n) = S(x + n)) □ (□x) (S(x) + S(n)) = S(x + S(n)))	□i 6, 20
22.	(((□x) (S(x) + 0) = S(x + 0)) □ (□n) (((□x) (S(x) + n) = S(x + n)) □ (□x) (S(x) + S(n)) = S(x + S(n))) □ (□n) (□x) (S(x) + n) = S(x + n)	A7
23.	(□n) (□x) (S(x) + n) = S(x + n)	□ e 21, 22
24.	y_1	
25.	(□x) (S(x) + y_1) = S(x + y_1)	□n e 23
26.	(□y) (□x) (S(x) + y) = S(x + y)	□y i 24-25

3. □ |□ (□x)(□y) (x + y) = (y + x)

1.	y_0	
2.	(0+ y_0) = y_0	□y e Example 1
3.	(y_0 + 0) = y_0	□x e A3
4.	(0+ y_0) = (y_0 + 0)	=e 2, 3
5.	(□y) (0 + y) = (y + 0)	□y i 1-5
6.	n	
7.	(□y) (n + y) = (y + n)	Assumption
8.	y_1	
9.	(n + y_1) = (y_1 + n)	□y e 7
10.	S(n + y_1) = S(n + y_1)	=i
11.	S(n + y_1) = S(y_1 + n)	=e 9, 10
12.	(□y) (n + S(y)) = S(n +y)	□x e A4
13.	y_1 + S(n) = S(y_1 + n)	□y e 12
14.	S(n + y_1) = y_1 + S(n)	=e 10, 13
15.	(□x) (S(x) + y_1) = S(x + y_1)	□y e Example 2
16.	S(n) + y_1 = S(n + y_1)	□x e 15
17.	S(n) + y_1 = y_1 + S(n)	=e 14, 16
18.	(□y) S(n) + y = y + S(n)	□y i 8-17
19.	(□y) (n + y) = (y + n) □ (□y) S(n) + y = y + S(n)	□ i 7-18
20.	(□n) (□y) (n + y) = (y + n) □ (□y) S(n) + y = y + S(n)	□n i 6-19
21.	(□y) (0 + y) = (y + 0) □ ((□n) (□y) (n + y) = (y + n) □ (□y) S(n) + y = y + S(n))	□i 5, 20
22.	((□y) (0 + y) = (y + 0) □ ((□n) (□y) (n + y) = (y + n) □ (□y) S(n) + y = y + S(n))) □ (□n) (□y) n + y = y + n))	A7
23.	(□n) (□y) n + y = y + n	□ e 21, 22
24.	x_1	
25.	(□y) x_1 + y = y + x_1	□n e 23
26.	(□x) (□y) x + y = y + x	□x i 24-25

Axioms for number theory:

- A1: (□x) □(S(x) = 0)
- A2: (□x) (□y) ((S(x) = S(y)) □ (x = y))
- A3: (□x) (x + 0) = x
- A4: (□x) (□y) (x + S(y)) = S(x+y)
- A5: (□x) (x * 0) = 0
- A6: (□x) (□y) (□z) (x*S(y)) = ((x*y) + x)
- A7: (□[0/x] □ (□n) (□[n/x] □ □[S(n)/x])) □ (□n)□[n/x]

  for **every formula** □ where n is free for x, for any variable x.