Propositional Natural Deduction

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Logic

- In CS 60 we had an introduction to both **proposition-** and **predicate-logic**.
- These were covered from the viewpoint of **meaning** (known as "model theory" to logicians).
- There is another part of the story dealing with the structure of **proofs** (known as "proof theory").
- We focus on the latter now, and will connect the two eventually.

Logic in CS 81

- We have two **objectives** in studying formal logic:
  - To firm up our concept of what forms a proof and how to create proofs.
  - To investigate the connection between computability and provability, such as:
    - The problem of giving an algorithm that will determine whether or not certain kinds of statements can be proved from certain axioms is **unsolvable**.

Formal Systems

- A system of logical proof is a variety of **formal system**, just as grammars and Turing machines are formal systems.
- A formal system tells how to **construct** things, using precise rules, usually as some form of induction.
- "Formal" means that adherence to the rules can be checked **algorithmically**.

Gottlob Frege (1848-1925)

- Created modern logic by introducing the predicate calculus.
- Developed a formalized definition of "proof".
- Defined the natural numbers in anticipation of Peano's axiomatization (1889)
- Did not anticipate Russell's paradox.

Varieties of Logical Proof Systems

- Axiomatic or Hilbert/Ackermann:
  - Basis is a set of **axioms**
  - Rules of inference tell how to derive **theorems** from axioms (in zero or more steps).
  - Relatively few rules of inference
- Natural Deduction, or Gentzen:
  - No axioms
  - Rules of inference tell how to derive **sequents**, which can entail axioms as pre-conditions and theorems as post-conditions.
  - Relatively many rules of inference.
- The two are equivalent; it is a matter of style.

Hilbert/Ackerman and Gentzen

- David Hilbert (1862-1943)

- Wilhelm Ackermann (1896-1962)
  student of Hilbert (no photo available)

- Gerhard Gentzen (1909-1945)

Natural Deduction

- A natural deduction system derives **sequents**, expressions of the form:

$$\square_1, \square_2, \ldots, \square_n \, |\square \; \square$$

- Each of the $\square_i$ and □ represents a **logical formula** in an appropriate language (in the sense we have been using the term).
- The interpretation of the sequent is that each $\square_i$ is a **premise** and □ is the **conclusion**.
- The $\square_i$ *could* be **axioms**, then □ would be a **theorem**. However, the word "theorem" is usually reserved for the case that the set of premises is empty.

Truth vs. Derivation

- The **intended interpretation** of the sequent is that □ is a **true** formula provided that each of the $\square_i$ are true.
- Whether or not this is really the case will depend on the rules.
- The definition of "**truth**" will be given later, but you can assume that it is like the one you know.
- Derivations themselves do not rely on notions of truth; they are totally **mechanical**.

Reference

- There are several approaches using sequents and different languages for formulas.
- We will be following the one in Huth & Ryan (HR).

A Typical Propositional Language

- E is the start symbol
- E □ A | // Atom
  (□E) | // Negation (not)
  (E □ E) | // Conjunction (and)
  (E E) | // Disjunction (or)
  (E '□ ' E) | // Implication (implies)
  □ | // Bottom
  T // Top
- A □ 'p' | 'q' | 'r' | 's' | … // Propositions

Bottom and Top?

- Think of bottom (□) as representing the constant "false".
- Think of top (T) as representing the constant "true".

Precedence

- The language as given fully parenthesizes everything.
- We will allow precedence in lieu of parentheses as an **abbreviation**. The binding order is negation, conjunction, disjunction, implication.

- So
  ((p □ (□q))   ((□r) □ (s □ q)))

  could be abbreviated:

  (p □ □q)   (□r □ (s □ q))

Examples of Sequents

- p, (p □ q) |□ q

- (p   q), □p |□ q

- (p □ q), (p   r), □r |□ q

- The first, for example, is interpreted “if p is true and (p □ q) is true, then q is true”.

More Notes on Sequents

- On the left-hand side of |□ in

  $\square_{1}$, $\square_{2}$, . . . $\square_{n}$ |□ □

  the formulas are regarded as a **set**:
  - order doesn’t matter
  - repetition doesn’t matter
- Order and repetition does matter **within** a formula. Formulas are just strings.

Sequents and Intuition

- You might be thinking “Why bother with sequents; I can do all of this with my knowledge of tautologies, etc.”

- Your knowledge can be used as **intuition** for validating a sequent.

- However, sequents are supposed to express whether certain **deductions** are valid, as they might occur in a mathematical proof.

- Tautologies won’t be enough when we introduce predicates and quantifiers.

- In addition to **using** sequents, we intend to **study** the **proof systems** themselves (called meta-logic).

Sequent Meta-Logical Issues

- **Soundness**:
  - Determine whether a sequent derives **only** true formulas from true formulas.
- **Completeness**:
  - Determine whether **every** true formula can be derived from a fixed set of formulas (axioms).

Natural Deduction Rules

- Each rule represents an **allowable step** in deriving a sequent.

- The rules focus on deriving formulas by **introducing** or **eliminating** the various connectives:
  □
  □

  □
- There is one rule for each case (introduction and elimination) for at least each connective, i.e. at least 8 rules. Some rules have multiple sub-rules.

Why "Natural" Deduction?

- "Natural" is a slogan intending to suggest that these rules are ones that might be used in normal proof construction and argumentation.
- Natural deduction also allows an argument to be developed by examining the desired conclusion and working toward assumed premises in a "natural" way.

□-Introduction Rule (□i)

- $\frac{□ \quad □}{□□□}$ (□i)
- The reading of this rule is:
  - If □ and □ are any formulas that follow from the premises of a sequent,

    then the formula □□□ also follows from those premises.
- The formulas above the line are called the **antecedents** and the one below the **consequent**.

Rule vs. Sequent

- Every rule immediately creates an infinite number of sequents. For example, the rule
  $\frac{□ \quad □}{□□□}$
  creates sequents **of the form**
  □, □ |□ (□□□)
  for every pair of formulas □ and □.
- The greek letters in the sequent form shown are not **the** formulas; they stand for aribtrary formulas.
- Many sequents require **multiple** rule applications to establish.

Examples of Sequents Derived Using Only the (□i) Rule

- p, (q r) |□ p□(q r) [One rule app.]
- p, (q r) |□ (q r)□p [One rule app.]
- p, (q r), s |□ ((q r)□(p□s)) [Two rule apps.]

Showing Sequent Derivations by Steps

- Derive p, (q r), s |□ ((q r)□(p□s)):

1.	p	Premise
2.	(q r)	Premise
3.	s	Premise
4.	(p□s)	Rule □i applied to formulas 1, 3
5.	((q r)□(p□s))	Rule □i applied to formulas 2, 4

- The numbers on the right refer to the **antecedents** used in the rule to obtain the formula on the left, which is the **consequent** of a rule.

Showing Sequent Derivations by DAGs

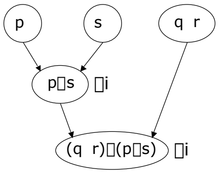
- DAG = "Directed Acyclic Graph"
- The premises are at the leaves of the DAG.

$$\frac{\frac{p \quad s}{(p□s)}\,□i \quad (q\ r)}{((q\ r)□(p□s))}\,□i$$

- Note that (p□s) is used as the consequent of one rule application and the antecedent of another.

DAG made more evident

$$\dfrac{\dfrac{p \quad s}{(p\wedge s)}\wedge i \qquad (q\ \ r)}{((q\ \ r)\wedge(p\wedge s))}\wedge i$$



Steps vs. DAGs

- Steps correspond to the way that an argument might be presented in a math text or paper.
- DAGs allow for better visulization of what is used for what.
- Either representation can be constructed from the other.

∧-Elimination Rule ($\wedge e_1$, $\wedge e_2$)

- $\dfrac{\phi\wedge\psi}{\phi}$ ($\wedge e_1$)
- $\dfrac{\phi\wedge\psi}{\psi}$ ($\wedge e_2$)
- Two sub-rules are needed because **order matters** within a formula. This rule eliminates one side of the ∧ or the other.

A Step Derivation Using ∧e and ∧i

- Derive p ∧ (q ∧ r) |− (p ∧ q) ∧ r :

1.	p ∧ (q ∧ r)	Premise
2.	p	$\wedge e_1$ 1
3.	q ∧ r	$\wedge e_2$ 1
4.	q	$\wedge e_1$ 3
5.	r	$\wedge e_1$ 3
6.	p ∧ q	∧i 2, 4
7.	(p ∧ q) ∧ r	∧i 6, 5

A DAG Derivation Using ∧e and ∧i

- Derive p ∧ (q ∧ r) |− (p ∧ q) ∧ r :

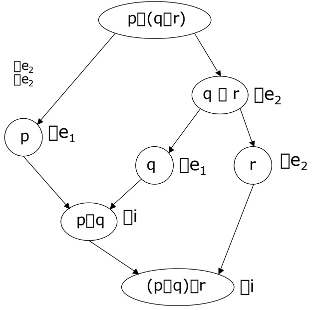
p ∧ (q ∧ r) $\wedge e_1$ $\wedge e_2$
p (q ∧ r) $\wedge e_1$ $\wedge e_2$
q r ∧i
(p ∧ q) ∧i
(p ∧ q) ∧ r

This shows that the DAG is not generally a “tree”, as some antecedents are used multiple times.

DAG made more evident

- Derive p ∧ (q ∧ r) |− (p ∧ q) ∧ r :

p ∧ (q ∧ r) $\wedge e_1$ $\wedge e_2$
p (q ∧ r) $\wedge e_1$ $\wedge e_2$
q r ∧i
(p ∧ q) ∧i
(p ∧ q) ∧ r



Constructing Proofs by Working Backward

- If the conclusion is a premise, there is nothing to do.
- Otherwise, the outermost logical connective may suggest what rule could be used:
  - Derive p □ (q □ r) |□ (p □ q) □ r
  - The outermost connective in the conclusion is □ therefore use □i as the last step:
    - (p □ q) □ r   □i 6, 5
  - The use of □i will require derivation of two new formulas:
    - (p □ q)   r
  - Apply this approach recursively.

Choices

- Often the rule choice is not unique.
- Make a choice, but be prepared to backtrack (crossing off what you have done) and try a different one.

Constructing Proofs by Working Forward

- If a premise is the conclusion, there is nothing to do.
- Otherwise, synthesize a formula from existing formulas using available rules.
- Working forward might entail many choices of a formula to be synthesized, not all of which will be useable in deriving the conclusion.

Constructing Proofs by Working Both Directions Simultaneously

- Blend together working backward with working forward until the two “meet in the middle”.
- Don’t overlook the DAG model as a means of arriving at proofs.
- Consider converting the DAG to steps for final clarity.

-Introduction Rule (i_1, i_2)

- $\dfrac{\square}{\square \quad \square}$ (i_1)
- $\dfrac{\square}{\square \quad \square}$ (i_2)

□ -Elimination Rule, Modus Ponens

- $\dfrac{\square,\ \square\square\ \square}{\square}$ (□ e)
- Its latin name ***modus ponens*** (**MP**) is often used for this rule.

Example using □ -Elimination Rule

- Derive p, (p □ q), (q □ r) |□ r

1. p Premise
2. p □ q Premise
3. q □ r Premise
4. q □ e 1, 2
5. r □ e 4, 3

- With this example, you can start to see how deriving a sequent might actually be easier (and more "natural") than establishing a tautology.

Another form of □ -Elimination Rule, Modus Tollens

- A related **macro** or "derived rule" is ***modus tollens*** (**MT**):
- $\frac{\square\square\ \square,\ \square\square}{\square\square}$ (MT)
- "macro" means that this rule is a convenience and can be treated as an abbreviation for the application of other rules.
- We will elaborate on this later.

Example using MT

- Derive □r, (p □ q), (q □ r) |□ □p

1. □r Premise
2. p □ q Premise
3. q □ r Premise
4. □q MT 3, 1
5. □p MT 2, 4

□□-Elimination and Introduction Rules

- $\frac{\square\square\square}{\square}$ (□□e)
- $\frac{\square}{\square\square\square}$ (□□i) (This rule is "derived".)

Rules with Sub-Derivations

- Certain rules have **sub-derivations**, rather than simply formulas, in their **antecedents**.
- A sub-derivation may incorporate **assumptions** that behave as **premises** but are not premises of the sequent being proved.
- These assumptions must be treated carefully to avoid confusion with regular premises.
- Accordingly, sub-derivations are shown inside a **box**.
- **Assumptions introduced inside the box cannot be used as premises outside the box.**
- **However**, sub-derivations **may** use formulas derived earlier **outside** the box.

$$\boxed{\begin{array}{c}\square\\ \cdot\\ \cdot\\ \cdot\\ \square\end{array}}$$

□ -Introduction Rule

- This is an example of a rule using a sub-derivation.

$$\frac{\boxed{\begin{array}{c}\square\\ \cdot\\ \cdot\\ \cdot\\ \square\end{array}}}{\square\square\ \square}\quad (\square\ i)$$

- Here to derive we use □ as an assumption and get □ as a conclusion using a sub-derivation.
- The sub-derivation is in a box because □ is not useable outside.

Example Using Sub-Derivation

- Derive (p □ q), (q □ r) |□ (p □ r)

1.	p □ q	Premise	
2.	q □ r	Premise	
3.	p	Assumption	
4.	q	□ e	1, 2
5.	r	□ e	2, 4
6.	p □ r	□ i	2-5

Another Example Using Sub-Derivation

- Derive (□p □ □q) |□ (q □ p) :

1.	□p □ □q	Premise	
2.	q	Assumption	
3.	□□q	□□i	2
4.	□□p	MT	1, 3
5.	p	□□e	4
6.	q □ p	□ i	2-5

- Pattern matching:
  □□ □, □□ (MT) □ is □p, □ is □q,
  □□ □□ is □□p, □□ is □□q

A Sub-Derivation can be Trivial

- Derive |□ (p □ p) (Set of premises is empty):

1.	p	Assumption	
2.	p □ p	□ i	1, 1

- Pattern matching:

□
.
.
.
□
(□ i)
□□ □ Both □ and □ are p.

Sub-Derivations can be Nested

- Derive (p □ q) □ r |□ p □ (q □ r)

1.	(p □ q) □ r	Premise
2.	p	Assumption
3.	q	Assumption
4.	p □ q	□i 2,3
5.	r	□ e 1, 4
6.	q □ r	□ i 3-5
7.	p □ (q □ r)	□ i 2-6

Sub-Derivations and DAGs

- It is unclear how to show sub-derivations in the DAG model.
- The customary way is to introduce the sub-derivation and **discharge** (**cross-out**) the assumptions so that they cannot be used outside the sub-derivation.
- The steps model is clearer in this regard, because nesting shows the order of discharge.

Sub-Derivations in the DAG model

- Derive (p □ q) □ r |□ p □ (q □ r)

$\cancel{p}$	Assumption (/ denotes discharged)
$\cancel{q}$	Assumption
(p □ q)	□i
(p □ q) □ r	Premise
r	□ e
q □ r	□ i
p □ (q □ r)	□ i

-Elimination Rule

- This rule uses two sub-derivations:

$$\frac{\square \quad \square \qquad \boxed{\begin{array}{c}\square\\ .\\ .\\ \square\end{array}} \quad \boxed{\begin{array}{c}\square\\ .\\ .\\ \square\end{array}}}{\square} \ (\ \ e)$$

- The interpretation is that if we want to "get rid of" a disjunction, we can derive a common formula from the two disjuncts.

Sub-Derivations vs. Sequents?

- Aren't the boxed sub-derivations essentially sequents themselves?
- If so, why don't we use the notation □ |□ □ rather than

$$\boxed{\begin{array}{c}\square\\ .\\ .\\ .\\ \square\end{array}}$$

- The answer probably lies in the fact that sub-derivations can make use of formulas **outside** the box, and we'd have to repeat those formulas as premises of the sequent.

□-Introduction Rule

This rule introduces □ through "contradiction":

$$\frac{\boxed{\begin{array}{c}\square\\ .\\ .\\ .\\ \square\end{array}}}{\square\square} \ (\square i)$$

□-Elimination Rule

$$\frac{\square \ \square \ \square\square}{\square} \qquad (\square e)$$

□-Elimination Rule

$$\frac{\square}{\square} \ (\square e)$$

- If we can derive □ then we can derive anything. Consequently, the things we derive won't have much information value. So being able to derive □ is undesirable, except in a sub-derivation.

Macro or Derived Rules

- Earlier MT was mentioned as a "macro" rule.
- The name "macro" alludes to programming language macros.
- While superficially similar to a subroutine, a macro is a text substitution done before a source is compiled or interpreted.
- In our case, it is a rule that could be replaced with a sequence of uses of other rules.

MT as a Macro derived from other rules

- □□ □, □□ (MT)
  □□

1.	□□ □	Premise
2.	□□	Premise
3.	□	Assumption
4.	□	□ e 1, 3
5.	□	□e 4, 2
6.	□□	□i 3-5

- Every use of MT could thus be replaced with this sequence, which uses 3 rules: □ e, □e, □i.

□□i as a Macro derived from other rules

- □ (□□i)
  □□□

1.	□	Premise
2.	□ □	Assumption
3.	□	□e 1, 2
4.	□□□	□i 2-3

Macro vs. Sequent

- Why isn't a macro rule just another sequent?

RAA (Reductio ad absurdum) Rule

- This rule has a similarity to □i:

□□
.
.
.
□
(RAA)
□

RAA as a Macro derived from other rules

1.	□□ . . . □	Premise
2.	□□ □ □	□ i 1
3.	□□	Assumption
4.	□	□e 2,3
5.	□□□	□i 3-4
6.	□	□□e 5

LEM (Law of the Excluded Middle)

- (No antecedent)
  □ □□

1.	□(□ □□)	Assumption
2.	□	Assumption
3.	□ □□	i_1 2
4.	□	□e 3, 1
5.	□□	□i 2-4
6.	□ □□	i_2 2
7.	□	□e 6, 1
8.	□ □(□ □□)	□i 1, 7
9.	□ □□	□□e 8

Summary of Non-Derived Rules

Connective	Introduction	Elimination
$\wedge$	$\wedge$i	$\wedge e_1$, $\wedge e_2$
$\vee$	$\vee i_1$, $\vee i_2$	$\vee$e
$\rightarrow$	$\rightarrow$ i	$\rightarrow$ e
$\neg$	$\neg$i	$\neg$e
$\bot$	(none)	$\bot$e
$\neg\neg$	(derived)	$\neg\neg$e

Summary of Derived Rules So Far

- MT (Modus Tollens)
- RAA (Reductio ad Absurdum)
- LEM (Law of the Excluded Middle)
- $\neg\neg$i

Validity vs. Provability

- $\phi_1, \ldots, \phi_n \vdash \psi$ means ψ is **provable** from $\phi_1, \ldots, \phi_n$
- $\phi_1, \ldots, \phi_n \models \psi$ means roughly the following:

  If each of ϕ_i is true, then ψ is true.
- In other words, ψ is a **valid** conclusion from $\phi_1, \ldots, \phi_n$.
- We need a definition of ***truth*** to make this precise.