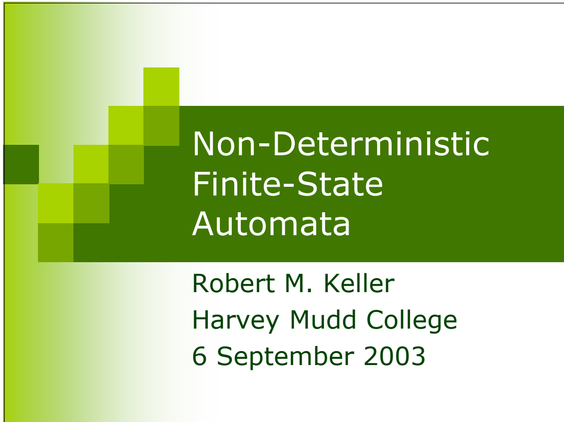


What is Non-Determinism?

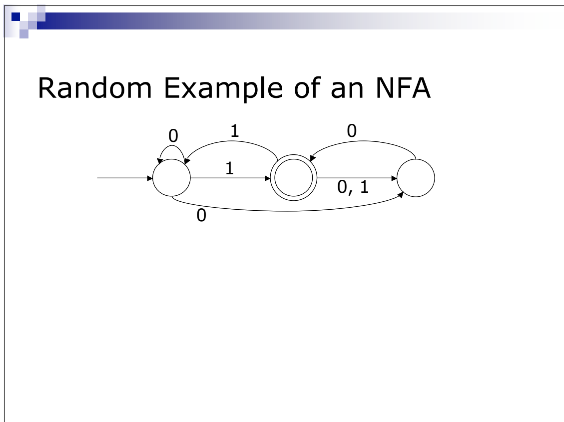
- Can an automaton have choice or free-will?
- Non-determinism supposes it can.

Uses of Non-Determinism?

- Modeling asynchronous events, wherein we don't know the order in which things happen.
- Modeling certain kinds of parallel computing.
- As a mathematical construction device.

NFA's

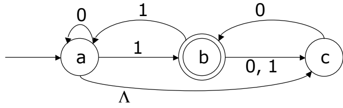
- NFA = Non-deterministic finite-state automaton
- From any given state, we can have **more than one** transition for the same letter.
- From any given state, we can have **no** transitions for a given letter.
- We can have free transitions that occur without using any letter. (These are labeled with Λ.)
- We can have multiple starting states, freely chosen.



Paths in an NFA

- Let x be a **string** over the alphabet of N.
- Let q and q' be states of N.
- We say there is a "**path from q to q' with label x**" provided that there is a series of arcs connecting q to q' and the labels on that series, when concatenated, form x.
- Note that Λ-transitions can be concatenated and do not change the string to which they are concatenated.

Paths in an NFA (not identical to previous)

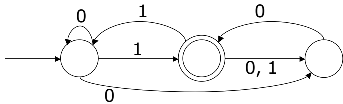


Some Paths
1 from a to b
Λ from a to c
0 from a to c
00 from a to b
11 from a to a
1 from b to c
010 from a to a
. . .

Acceptance by an NFA

- An NFA accepts a string x if there is **a** path from **some** start state to **some** accepting state with label x.
- If there is no such path, then the NFA does not accept the string.
- The **language** accepted by an NFA is the set of all strings accepted by the NFA.

Original Example of an NFA



Strings accepted:	Strings not accepted:
1	Λ
00	0
01	10
000	11
001	010
100	011
110	101
111	. . .
. . .	

Observation

- A DFA can be viewed as a special case of an NFA.
- If L is a language accepted by a DFA, then L is also accepted by a NFA.

The NFA Theorem

- If L is accepted by an NFA, there is also a DFA that accepts L.

Proof of the NFA Theorem (called the "Subset Construction")

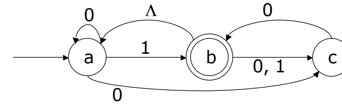
- Suppose we have an NFA N. We want to construct a DFA D that accepts the same language exactly.
- Let S be the set of states of N. Choose as the states of D the set P(S) = the **set of all subsets** of the states of N. Since S is finite, so is P(S).
- (In practice, we won't usually need **all subsets**.)

Transitions of D

- Let R be a state of D. So $R \subseteq S$.
- We must define a transition from R for each symbol σ in our alphabet.
- To designate the **next state** from R given symbol σ, use:

$$R_\sigma = \{q' \in S \mid (\exists q \in R)\ \exists \text{ path from } q \text{ to } q' \text{ with label } \sigma\}$$

Examples of DFA Transitions



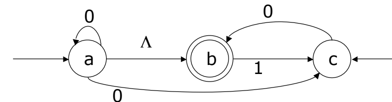
$\{a, b\}_0 = \{a, c\}$
$\{a, b\}_1 = \{a, b, c\}$
$\{b\}_0 = \{a, c\}$
$\{b\}_1 = \{c\}$
$\{c\}_0 = \{a, b\}$
$\{c\}_1 = \varnothing$

Initial State of D

- Letting S_0 be the set of initial states of N, the initial state of D is defined to be the subset

$$\{q' \in S \mid (\exists q \in S_0)\ \exists \text{ path from } q \text{ to } q' \text{ with label } \Lambda\}$$

Examples of Initial State



The initial state of D is {a, b, c}.

Conclusion of the NFA Theorem

- The preceding construction shows how we capture all paths of the original NFA N in a DFA D.
- The only part remaining is defining the **accepting states** of D. These will be the subsets of S that **contain at least one accepting state** of N.

Interpretation of the DFA Construction

- The DFA constructed in the proof of the theorem can be interpreted as an automaton that simulates all possible choices of the NFA **in parallel**.
- This idea will have at least one other application (to be seen).

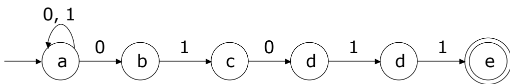
A Computational Shortcut

- By beginning with the initial state of the DFA and working from there, we can eliminate the need to generate all states in P(S) in most cases.
- In other words, we need only generate the states **reachable from** the initial state of D.

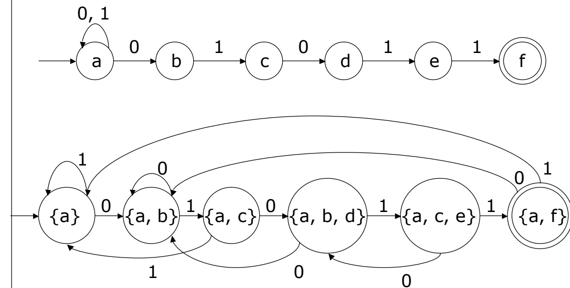
Application of the Construction

- Suppose we want to construct a DFA that will accept the language of **strings ending with a given string**, for example: 01011.
- The tricky part here is that if we get as input, for example, 01010, while this is not accepted, the last part 010 can possibly be used as the initial part of a string that is, for example 0101011.
- By constructing an appropriate NFA and converting it, it is easy to get the DFA right.

An NFA for {0, 1}* 01011



Constructing a DFA for {0, 1}* 01011



Notes on the Previous Example

- Not all subsets were reachable. Those that weren't were not generated.
- The number of states is the same. This is a coincidence; it may be more or fewer.
- The transition structure is more complex in the DFA. This is typical.

Algorithmic Application

- The principle underlying the illustrated method for text matching was the insight and basis for two text searching algorithms:
  - **Knuth-Morris-Pratt**: Search for a single string
  - **Aho-Corasick**: Search for a finite set of strings

Another Possibility for Search

- Rather than go through the DFA construction and simulate the result to do search a text, it is possible to simulate the NFA directly.
- In this case there would be a **set** of "current states" rather than a single one.
- Just use **multiple state pointers** for the simulation rather than a single one.

More applications of the subset construction are forthcoming.

- Closure properties
- DFA from regular expressions

Families of Languages

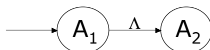
- Let F be a **family** (set) of languages.
- Examples:
  - The family of finite languages.
  - The family of regular languages.
  - The family of co-finite languages (complement within Σ^* is finite).
  - The family of all languages.

Closure Properties

- A family F is **closed under** an operator if the application of that operator to a language or languages in F results in a language which is also in F.
- Example: The family of regular languages is closed under $\cup$, concatenation, and *.
- Exercise: Determine whether the other families on the preceding page are closed under these same operators. (Create a matrix.)

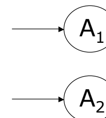
Proof of Some Closure Properties using the Subset Construction

- The family of languages accepted by DFA (or NFA; we know they are the same) is closed under **concatenation**.
- **Proof**: Suppose L_1 and L_2 are accepted by DFA's. To show that $L_1 L_2$ is also, connect the corresponding acceptors A_1 and A_2 by adding Λ transitions from each accepting state of A_1 to each start state of A_2. Then modify the start and accepting states appropriately and convert this NFA to a DFA. ***Figuratively***,



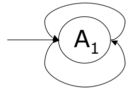
Proof of Some Closure Properties using the Subset Construction

- Show that the family of DFA languages is closed under union. Hint:



Proof of Some Closure Properties using the Subset Construction

- Show that the family of DFA languages is closed under *. Hint:



Summary

- The family of languages accepted by DFA's is closed under concatenation, union, and *.

Proof that every regular language is accepted by a DFA.

- The family of languages accepted by DFA's is closed under concatenation, union, and *. These correspond exactly to the regular operators.
- Furthermore, the languages of a single 1–letter string are accepted by DFA's, as are ∅ and Λ.
- Therefore every regular language is accepted by a DFA.

Kleene's Theorem

- A language is regular iff it is accepted by some DFA.
- Proof:
  - We showed DFA ⇒ regular by solving a system of equations.
  - We showed regular ⇒ DFA by the subset construction and closure properties.

The family of DFA languages is closed under complement.

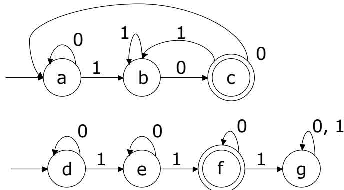
- For complement, we only need reverse the roles of accepting and non-accepting states in the accepting automaton.
- Thus the family of regular languages is closed under complement, even though complement is not a regular operator. (It is sometimes seen in an **extended** version of regular expressions.)

The family of DFA languages is closed under intersection.

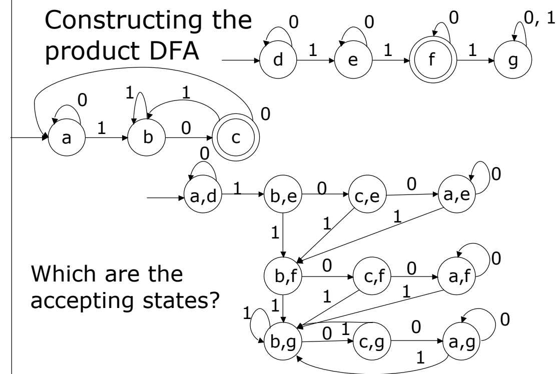
- Proof: Adjoin the two automata and use the subset construction with the set of two start states as the start state. This effectively constructs an automaton, called the **product automaton**, that simulates the behavior of both of the original automata in parallel.
- Choose accepting states appropriately. The new automaton will automatically be deterministic, as the originals were.
- Thus the family of regular languages is closed under intersection, although ∩ is not a regular operator.
- The product automaton can be used for any binary set operator (∩, ∪, -, ⊕, ≡, etc.). Only the accepting states are different.

Example: A DFA for the intersection of languages given by regular expressions (0∪1)*10 and 0*10*10*

The individual DFA are:



Constructing the product DFA



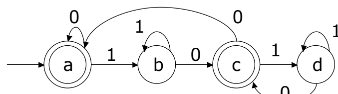
Notes on Product Construction

- Depending on the intended implementation, it may be better **not** to construct the composite machine explicitly, but rather leave it decomposed.
- The rationale is that there are generally fewer states in the sum of the two machines than in the composite.
- We sometimes try to go the other way: **decompose** a complex machine into a product of simpler machines.

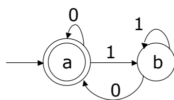
Equivalence of States

- For any state, not just the start state, we can define a **language accepted from** that state.
- Two states are equivalent if their languages are equal.
- If an automaton contains distinct equivalent states, those states can be “merged” to get a simplified automaton with fewer states.

Example of State Equivalence



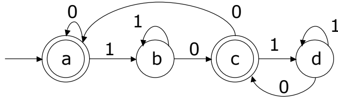
In this example, state a is equivalent to state c and state b is equivalent to state d. The simplified automaton is:

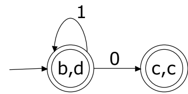


Equivalence Testing

- How can we test whether two states are equivalent or not?
- One way: Think of ≡ as a binary operator, just like ∩, ∪, etc. Construct the product machine for two automata, one having each of the two states as starting states. Accepting states are those with both pairs accepting or both pairs non-accepting.
- The two states are equivalent iff the product machine accepts Σ^*.

Example: Equivalence Testing of b and d:





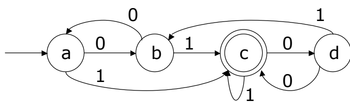
Note: Once we get to a state where both components are the same, we don't need to complete that branch of the construction because the graph will be that of the original machine.

Conclusion: b and d are equivalent.

State Partitioning Algorithm

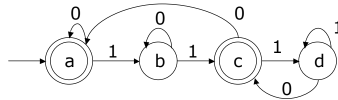
- This algorithm finds the equivalence classes of states in a DFA.
- Start with a partition P of the states into accepting and non-accepting.
- Repeat until there is no change in P:
  - For each element of P, determine whether for each input symbol the next states are in the same element of P. If not, partition the element into states having the same next-state partition. Replace P with the result of partitioning.

State Partitioning Example 1



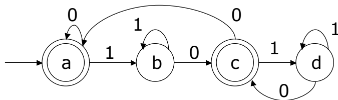
- P = {{a, b, d}, {c}}
- Next step: P = {{a, b}, {c}, {d}}, since b:1 = c but d:1 = b and b and c are in different elements.
- Now there can be no change.

State Partitioning Example 2



- P = {{a, c}, {b, d}}
- Next step: P = {{a, c}, {b}, {d}}, since b:1 = c but d:1 = d and c and d are in different elements.
- Next step: P = {{a}, {c}, {b}, {d}}, since a:1 = b but c:1 = d, and b and d are in different elements.
- Now there can be no change. No two states are equivalent.

State Partitioning Example 3



- P = {{a, c}, {b, d}}
- Each element of P has the indicated next-state property, so the algorithm terminates.
- a is equivalent to c, b is equivalent to d.

Comments on State Partitioning Algorithm

- The cases where there is only one set in the partition, or where all sets in the partition are singletons, will not change.
- We don't need to check singleton sets for splitting.
- Sets only split, never merge back together.
- At each step of the algorithm, there must be a split. Otherwise the algorithm terminates. Therefore, the algorithm requires at most n-1 steps for an n-state machine.
- The size of the partition at the start of the ith (i = 1, 2, ...) step must be at least i+1 for the algorithm to have another step.

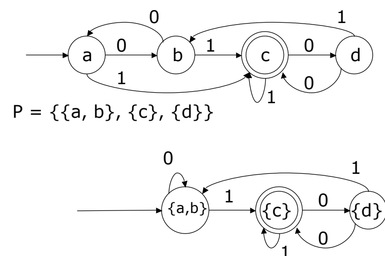
Examples of How States Can Partition

- Shortest: $\{\{s_1, s_2, \ldots s_n\}\}$ No further splitting possible.
- Nominal:
  $\{\{s_1, s_2, \ldots s_{n-1}\}, \{s_n\}\}$ 1st step
  $\{\{s_1, s_2\}, \{s_3, s_4\}, \ldots, \{s_{n-1}\}, \{s_n\}\}$ 2nd step
  ...
  $\{\{s_1, s_2\}, \{s_3\}, \{s_4\}, \ldots, \{s_{n-1}\}, \{s_n\}\}$ (n/2)nd step
  $\{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}, \ldots, \{s_{n-1}\}, \{s_n\}\}$ (1+n/2)nd step
- Longest:
  $\{\{s_1, s_2, \ldots s_{n-1}\}, \{s_n\}\}$ 1st step
  $\{\{s_1, s_2, \ldots s_{n-2}\}, \{s_{n-1}\}, \{s_n\}\}$ 2nd step
  $\{\{s_1, s_2, \ldots s_{n-2}\}, \{s_{n-2}\}, \{s_{n-1}\}, \{s_n\}\}$ 3rd step
  ...
  $\{\{s_1\}, \{s_2\}, \ldots, \{s_{n-2}\}, \{s_{n-1}\}, \{s_n\}\}$ (n-1)th step

Finding the Minimum State Acceptor

- Once equivalent states have been determined, an equivalent acceptor with the minimum number of states can be constructed.
- Simply let the **equivalence classes** of the original machine **be the states** of the new machine. Define transitions between such states to be consistent with those in the original. Define acceptance to be those classes containing accepting states.

State Minimization Example



Abstract States of a Language

- By looking at a language, we can tell something about what kind of a machine is required to recognize it.
- For any string x, the **abstract state** of language L for x is the set $L/x = \{y \mid xy \in L\}$.
- If this set is the same for two strings x and x', then it is ok for the machine to **share the same state** with both inputs x and x'. The "past history" doesn't matter.
- Otherwise, the machine must **distinguish** x and x' by being in different states after those strings are input.

Examples of L/x

- Suppose $L = \{01\}^*\{10\}$

  $L/1 = \{0\} = L/011 = L/01011 = \ldots$

  $L/10 = \{\Lambda\}$

  $L/0 = \{1\}\{01\}^*\{10\}$

  $L/00 = \varnothing = L/11 = \ldots$

  $L/\Lambda = \{01\}^*\{10\} = L/01 = L/0101 = \ldots$

The Abstract Machine for L

- The abstract states can be used to form an abstract machine. For each string x and letter σ there is a transition from L/x to $L/x\sigma$ with label σ.
- Example: Suppose $L = \{01\}^*10$

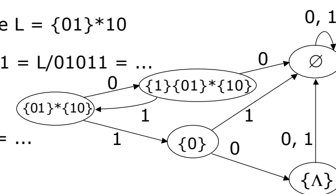
  $L/1 = \{0\} = L/011 = L/01011 = \ldots$

  $L/10 = \{\Lambda\}$

  $1\{01\}^*10 = L/0 = \ldots$

  $\varnothing = L/00 = \ldots$

  $\{01\}^*10 = L/\Lambda = L/01 = L/0101 = \ldots$



Distinguishability

- Two strings x and x′ are called **indistinguishable** under L, written
  $x \equiv_L y$
  provided
  $L/x = L/x'$.
- Two strings are **distinguishable** iff they are not indistinguishable.

Properties of $\equiv_L$

- $\equiv_L$ is an **equivalence relation**
- $\equiv_L$ is **right-invariant**, meaning:

  $x \equiv_L y$ implies $(\forall z)\ xz \equiv_L yz$

Note on Right-Invariance of $\equiv_L$

- The following are equivalent:
  1. $x \equiv_L y$ implies $(\forall z)\ xz \equiv_L yz$
  2. $x \equiv_L y$ implies $(\forall \sigma)\ x\sigma \equiv_L y\sigma$
- That $1 \Rightarrow 2$ is obvious.

  That $2 \Rightarrow 1$ requires string induction on z.

Theorem (Myhill-Nerode)

- L is accepted by some DFA iff $\equiv_L$ has a finite number of equivalence classes.
- Proof: This theorem is a summarization of the preceding discussion.

A Language that is not DFA-acceptable

- $L = \{0^n 1^n \mid n \in \omega\}$
- Abstract states L/0, L/00, L/000, . . . are distinct, and there is an infinite number of them.
- The reason that they are distinct is that $L/0^n$ contains 1^m iff $n = m$.

Summary

- The following are equivalent for a language L:
  - There is a DFA accepting L.
  - There is an NFA accepting L.
  - L is regular.
  - The equivalence relation $\equiv_L$ has a finite number of equivalence classes.
- Regarding the fourth point, the classes are effectively the states of the minimal DFA.