

# Natural Deduction Proof Examples

Robert Keller  
25 November 2003

# HR 2.5.1d

- Derive  $(\forall x)(P(x) \supset Q(x)) \mid \supset (\forall x) P(x) \supset (\forall x) Q(x)$  :

1.	$(\forall x)(P(x) \supset Q(x))$	Premise
2.	$x_0$	
3.	$P(x_0) \supset Q(x_0)$	$\forall x e 1$
4.	$P(x_0)$	$\forall e_1 3$
5.	$(\forall x) P(x)$	$\forall i 2-4$
6.	$x_1$	
7.	$P(x_1) \supset Q(x_1)$	$\forall x e 1$
8.	$Q(x_1)$	$\forall e_2 7$
9.	$(\forall x) Q(x)$	$\forall i 6-8$
10.	$(\forall x) P(x) \supset (\forall x) Q(x)$	$\forall i 5,9$

# HR 2.5.1e

- Derive  $(\forall x)P(x) \wedge (\forall x)Q(x) \mid\mid (\forall x) (P(x) \wedge Q(x))$  :

1.	$(\forall x)P(x) \wedge (\forall x)Q(x)$	Premise
2.	$x_0$	
3.	$(\forall x)P(x)$	Assumption
4.	$P(x_0)$	$\forall x e 3$
5.	$P(x_0) \wedge Q(x_0)$	$i_1 5$
6.	$(\forall x)Q(x)$	Assumption
7.	$Q(x_0)$	$\forall x e 5$
8.	$P(x_0) \wedge Q(x_0)$	$i_2 7$
9.	$P(x_0) \wedge Q(x_0)$	$e 1, 3-5, 6-8$
10.	$(\forall x) (P(x) \wedge Q(x))$	$\forall x i 2-9$

# HR 2.5.4b

- Derive  $(\forall x) (\forall y) F(x, y) \mid \vdash (\forall u) (\forall v) F(u, v)$  :

1.	$(\forall x) (\forall y) F(x, y)$	Premise
2.	$x_0$ $(\forall y) F(x_0, y)$	
3.	$y_0$ $F(x_0, y_0)$	
4.	$(\forall v) F(x_0, v)$	$\forall v$ i 5
5.	$(\forall v) F(x_0, v)$	$\forall y$ e 2, 3-4
6.	$(\forall u)(\forall v) F(u, v)$	$\forall v$ i 6
7.	$(\forall u)(\forall v) F(u, v)$	$\forall x$ e 1, 2-6

# HR 2.5.7f

- Derive  $\neg(\forall x)P(x) \mid \neg(\exists x)(\neg P(x))$  :

1.	$\neg(\forall x)P(x)$	Premise
2.	$x_0$	
3.	$P(x_0)$	Assumption
4.	$(\forall x)P(x)$	$\forall x i \ 3$
5.	$\perp$	$\perp e \ 4, 1$
6.	$\neg P(x_0)$	$\neg i \ 3-5$
7.	$(\exists x)(\neg P(x))$	$\exists x i \ 2-6$

# HR 2.5.11d

- Derive  $(\forall x) (\forall y) (S(x, y) \supset S(y, x)) \vdash (\forall x) (\forall y) S(x, y)$  :

1.	$(\forall x) (\forall y) (S(x, y) \supset S(y, x))$	Premise
2.	$x_0$ $(\forall y) (S(x_0, y) \supset S(y, x_0))$	Assumption
3.	$y_0$ $S(x_0, y_0) \supset S(y_0, x_0)$	Assumption
4.	$S(x_0, y_0)$	Assumption
5.	$(\forall y) S(x_0, y)$	$\forall y$ i 4
6.	$(\forall x)(\forall y) S(x, y)$	$\forall x$ i 6
7.	$S(y_0, x_0)$	Assumption
8.	$(\forall y) S(y_0, y)$	$\forall y$ i 7
9.	$(\forall x)(\forall y) S(x, y)$	$\forall x$ i 8
10.	$(\forall x)(\forall y) S(x, y)$	e 3, 4-6, 7-9
11.	$(\forall x)(\forall y) S(x, y)$	$\forall y$ e 2, 3-10
12.	$(\forall x)(\forall y) S(x, y)$	$\forall x$ e 1, 2-11



# Quantifier Proof Rules (for reference)



## $(\forall x)$ -Elimination Rule $(\forall x e)$

- $$\frac{(\forall x) \phi}{\phi[t/x]} \quad (\forall x e)$$

where  $t$  is any term that is free for  $x$  in  $\phi$ .

- What the rule says:**

If we have derived a universally-quantified formula  $\phi$ , then the formula  $\phi$  with any (appropriately-qualified) **specific instance** of  $x$  substituted for  $x$  is derivable.

# $(\Box x)$ -Introduction Rule

- This rule uses a sub-derivation, with **no formula assumed**.

$$\frac{\begin{array}{|l} x_0 \\ \cdot \\ \cdot \\ \cdot \\ \Box[x_0/x] \end{array}}{(\Box x)\Box} \quad (\Box x \text{ i})$$

- Here  $x_0$  is a “fresh” variable otherwise unused in the proof.
- $x_0$  must be free for  $x$  in  $\Box$ , but since  $x_0$  is “fresh”, this should never be an issue.



## $(\forall x)$ -Introduction Rule

- **What this rule says:**
- If we have argued to derive a term  $\square[x_0/x]$  where  $x_0$  is an **arbitrary** value of  $x$ , then we are justified in concluding  $(\forall x)\square$ .
- The key is the word “arbitrary”; there can be no constraints attached to  $x_0$ .
- Note: Once the conclusion  $(\forall x)\square$  is drawn,  $x_0$  is **discharged** and cannot be further used.



## $(\exists x)$ -Introduction Rule $(\exists x i)$

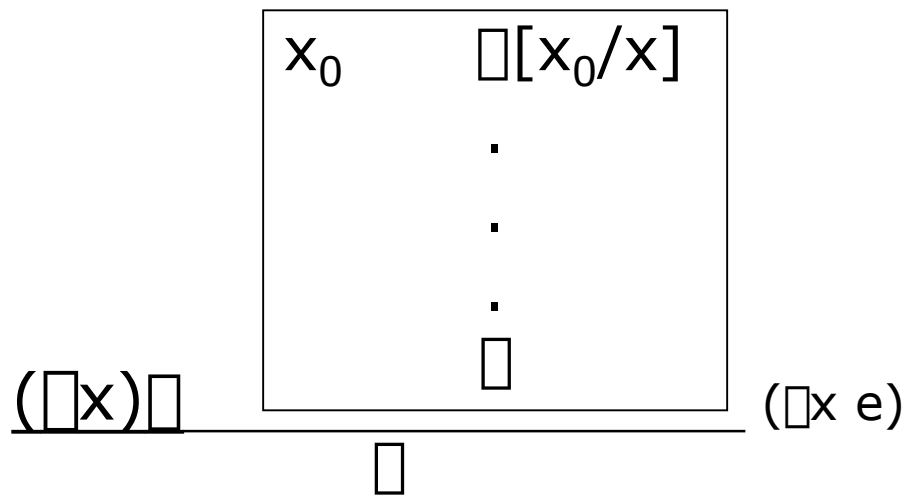
- $$\frac{\varphi[t/x] \quad (\exists x i)}{(\exists x) \varphi}$$

where  $t$  is any term that is free for  $x$  in  $\varphi$ .

- **What the rule says:**

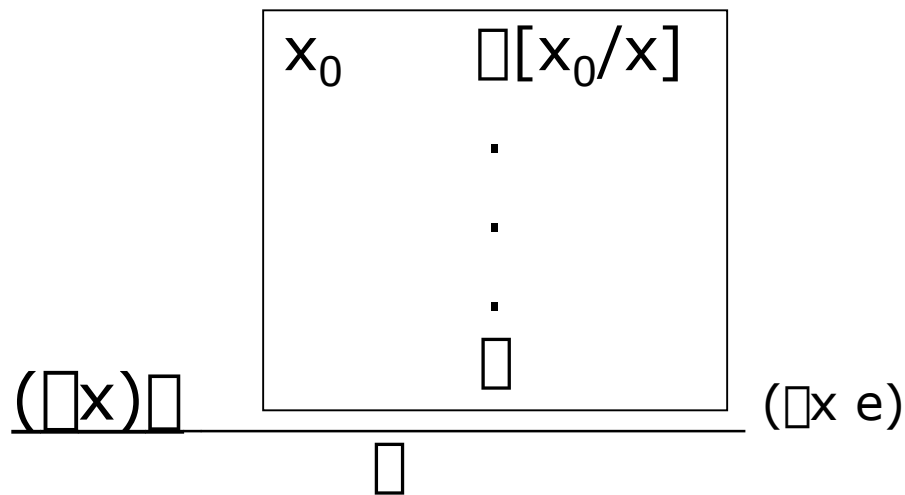
If we have exhibited a formula  $\varphi$  in which variable  $x$  is replaced by a **specific instance** then we can conclude that there is **an**  $x$  for for which the formula is true.

# $(\exists x)$ -Elimination Rule $(\exists x e)$



- Here  $x_0$  is a “fresh” variable otherwise unused in the proof.
- $x_0$  must be free for  $x$  in  $\Box$ , but since  $x_0$  is “fresh”, this should never be an issue.

## $(\exists x)$ -Elimination Rule $(\exists x e)$



- **What this rule says:**
- Assume that we have derived  $(\exists x)\varphi$ . One use we can make of this fact is to let  $x_0$  be **an**  $x$  such that  $\varphi[x_0/x]$ . There can be no other constraints on  $x_0$ . If we then derive  $\varphi$  from the assumption about  $\varphi$ , then we can conclude  $\varphi$  in general.