Natural Deduction Proof Examples

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HR 2.5.1d

- Derive $(\forall x)(P(x) \land Q(x)) \vdash (\forall x)\, P(x) \land (\forall x)\, Q(x)$:

1.		$(\forall x)(P(x) \land Q(x))$	Premise
2.	x_0		
3.		$P(x_0) \land Q(x_0)$	$\forall x$ e 1
4.		$P(x_0)$	$\land e_1$ 3
5.		$(\forall x)\, P(x)$	$\forall$i 2-4
6.	x_1		
7.		$P(x_1) \land Q(x_1)$	$\forall x$ e 1
8.		$Q(x_1)$	$\land e_2$ 7
9.		$(\forall x)\, Q(x)$	$\forall$i 6-8
10.		$(\forall x)\, P(x) \land (\forall x)\, Q(x)$	$\land$i 5,9

HR 2.5.1e

- Derive (□x)P(x)   (□x)Q(x) |□ (□x) (P(x)   Q(x)) :

1.	(□x)P(x) (□x)Q(x)	Premise
2. x_0		
3.	(□x)P(x)	Assumption
4.	$P(x_0)$	□x e 3
5.	$P(x_0)$ $Q(x_0)$	i_1 5
6.	(□x)Q(x)	Assumption
7.	$Q(x_0)$	□x e 5
8.	$P(x_0)$ $Q(x_0)$	i_2 7
9.	$P(x_0)$ $Q(x_0)$	e 1, 3-5, 6-8
10.	(□x) (P(x) Q(x))	□x i 2-9

HR 2.5.4b

- Derive (□x) (□y) F(x, y) |□ (□u) (□v) F(u, v) :

1.		(□x) (□y) F(x, y)	Premise
2.	x_0	(□y) F(x_0, y)	
3.	y_0	F(x_0, y_0)	
4.		(□v) F(x_0, v)	□v i 5
5.		(□v) F(x_0, v)	□y e 2, 3-4
6.		(□u)(□v) F(u, v)	□v i 6
7.		(□u)(□v) F(u, v)	□x e 1, 2-6

HR 2.5.7f

- Derive $\neg(\exists x)P(x) \vdash (\forall x)(\neg P(x))$:

1.	$\neg(\exists x)P(x)$	Premise
2.	x_0	
3.	$P(x_0)$	Assumption
4.	$(\exists x)P(x)$	$\exists x$ i 3
5.	$\bot$	$\neg$e 4, 1
6.	$\neg P(x_0)$	$\neg$i 3-5
7.	$(\forall x)(\neg P(x))$	$\forall x$ i 2-6

HR 2.5.11d

- Derive (□x) (□y) (S(x, y)   S(y, x)) |□ (□x) (□y) S(x, y) :

1.		(□x) (□y) (S(x, y) S(y, x))	Premise
2.	x_0	(□y) (S(x_0, y) S(y, x_0))	Assumption
3.	y_0	S(x_0, y_0) S(y_0, x_0))	Assumption
4.		S(x_0, y_0)	Assumption
5.		(□y) S(x_0, y)	□y i 4
6.		(□x)(□y) S(x, y)	□x i 6
7.		S(y_0 , x_0)	Assumption
8.		(□y) S(y_0, y)	□y i 7
9.		(□x)(□y) S(x, y)	□x i 8
10.		(□x)(□y) S(x, y)	e 3, 4-6, 7-9
11.		(□x)(□y) S(x, y)	□y e 2, 3-10
12.		(□x)(□y) S(x, y)	□x e 1, 2-11

Quantifier Proof Rules (for reference)

(x)-Elimination Rule (x e)

- $\dfrac{(\text{}x)\ \text{}}{\text{}[t/x]}$ (x e)

  where t is any term that is free for x in .

- **What the rule says:**

  If we have derived a universally-quantified formula , then the formula with any (appropriately-qualified) **specific instance** of x substituted for x is derivable.

$(\forall x)$-Introduction Rule

- This rule uses a sub-derivation, with **no formula assumed**.

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ \varphi[x_0/x] \end{array}}}{(\forall x)\varphi} \quad (\forall x\ i)$$

- Here x_0 is a “fresh” variable otherwise unused in the proof.
- x_0 must be free for x in φ, but since x_0 is “fresh”, this should never be an issue.

(□x)-Introduction Rule

- **What this rule says:**

- If we have argued to derive a term □$[x_0/x]$ where x_0 is an **arbitrary** value of x, then we are justified in concluding (□x)□.

- The key is the word “arbitrary”; there can be no constraints attached to x_0.

- Note: Once the conclusion (□x)□ is drawn, x_0 is **discharged** and cannot be further used.

(□x)-Introduction Rule (□x i)

- $\dfrac{\square[t/x]}{(\square x)\ \square}$ (□x i)

  where t is any term that is free for x in □.

- **What the rule says:**

  If we have exhibited a formula □ in which variable x is replaced by a **specific instance** then we can conclude that there is **an** x for for which the formula is true.

(□x)-Elimination Rule (□x e)

x_0	$□[x_0/x]$
	.
	.
	.
	□

(□x)□ ——————————— (□x e)

□

- Here x_0 is a "fresh" variable otherwise unused in the proof.
- x_0 must be free for x in □, but since x_0 is "fresh", this should never be an issue.

(□x)-Elimination Rule (□x e)

x_0	□[x_0/x]
	.
	.
	.
	□

(□x)□ ⟶ □ (□x e)

- **What this rule says:**
- Assume that we have derived (□x)□. One use we can make of this fact is to let x_0 be ***an*** x such that □[x_0/x]. There can be no other constraints on x_0. If we then derive □ from the assumption about □, then we can conclude □ in general.