HR 2.5.1d

Derive \((\forall x)(P(x) \supset Q(x)) \vdash (\forall x) P(x) \supset (\forall x) Q(x)\):

1. \((\forall x)(P(x) \supset Q(x))\) [Premise]
2. \(x_0\)
3. \(P(x_0) \supset Q(x_0)\) [\(\forall x \in 1\)]
4. \(P(x_0)\) [\(\forall e_1 3\)]
5. \((\forall x) P(x)\) [\(\forall i 2-4\)]

6. \(x_1\)
7. \(P(x_1) \supset Q(x_1)\) [\(\forall x \in 1\)]
8. \(Q(x_1)\) [\(\forall e_2 7\)]
9. \((\forall x) Q(x)\) [\(\forall i 6-8\)]
10. \((\forall x) P(x) \supset (\forall x) Q(x)\) [\(\forall i 5,9\)]
HR 2.5.1e

Derive \( (\exists x)P(x) \ (\exists x)Q(x) \vdash (\exists x) (P(x) \ Q(x)) \):

1. \( (\exists x)P(x) \ (\exists x)Q(x) \) Premise

2. \( x_0 \)

3. \( (\exists x)P(x) \) Assumption

4. \( P(x_0) \) \( \exists x \in 3 \)

5. \( P(x_0) \ Q(x_0) \) \( i_1 \ 5 \)

6. \( (\exists x)Q(x) \) Assumption

7. \( Q(x_0) \) \( \exists x \in 5 \)

8. \( P(x_0) \ Q(x_0) \) \( i_2 \ 7 \)

9. \( P(x_0) \ Q(x_0) \) \( \exists x \in 1, 3-5, 6-8 \)

10. \( (\exists x) (P(x) \ Q(x)) \) \( \exists x \in 2-9 \)
HR 2.5.4b

- Derive $(\forall x) (\forall y) F(x, y) |\Box (\forall u) (\forall v) F(u, v)$:

<table>
<thead>
<tr>
<th>Step</th>
<th>Expression</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$(\forall x) (\forall y) F(x, y)$</td>
<td>Premise</td>
</tr>
<tr>
<td>2.</td>
<td>$x_0$ $(\forall y) F(x_0, y)$</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>$y_0 F(x_0, y_0)$</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>$(\forall v) F(x_0, v)$</td>
<td>$\forall v i 5$</td>
</tr>
<tr>
<td>5.</td>
<td>$(\forall v) F(x_0, v)$</td>
<td>$\forall y e 2, 3-4$</td>
</tr>
<tr>
<td>6.</td>
<td>$(\forall u) (\forall v) F(u, v)$</td>
<td>$\forall v i 6$</td>
</tr>
<tr>
<td>7.</td>
<td>$(\forall u) (\forall v) F(u, v)$</td>
<td>$\forall x e 1, 2-6$</td>
</tr>
</tbody>
</table>
**HR 2.5.7f**

- Derive \( \Box(\forall x)P(x) \vdash (\forall x)(\Box P(x)) \):

<table>
<thead>
<tr>
<th>Step</th>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \forall x)P(x) )</td>
<td>Premise</td>
</tr>
<tr>
<td>2.</td>
<td>( x_0 )</td>
<td>Assumption</td>
</tr>
<tr>
<td>3.</td>
<td>( P(x_0) )</td>
<td>Assumption</td>
</tr>
<tr>
<td>4.</td>
<td>( \forall x)P(x) )</td>
<td>( \forall x ) i 3</td>
</tr>
<tr>
<td>5.</td>
<td>( \Box P(x_0) )</td>
<td>( \Box ) e 4, 1</td>
</tr>
<tr>
<td>6.</td>
<td>( (\forall x)(\Box P(x)) )</td>
<td>( \forall x ) i 3-5</td>
</tr>
<tr>
<td>7.</td>
<td>( (\forall x)(\Box P(x)) )</td>
<td>( \forall x ) i 2-6</td>
</tr>
</tbody>
</table>
**HR 2.5.11d**

Derive \((\forall x)(\forall y) (S(x, y) \land S(y, x)) \rightarrow (\forall x)(\forall y) S(x, y)\):

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((\forall x)(\forall y) (S(x, y) \land S(y, x)))</td>
<td>Premise</td>
</tr>
<tr>
<td>2</td>
<td>(x_0) ((\forall y) (S(x_0, y) \land S(y, x_0)))</td>
<td>Assumption</td>
</tr>
<tr>
<td>3</td>
<td>(y_0) (S(x_0, y_0) \land S(y_0, x_0))</td>
<td>Assumption</td>
</tr>
<tr>
<td>4</td>
<td>(S(x_0, y_0))</td>
<td>Assumption</td>
</tr>
<tr>
<td>5</td>
<td>((\forall y) S(x_0, y))</td>
<td>(\forall y) i 4</td>
</tr>
<tr>
<td>6</td>
<td>((\forall x)(\forall y) S(x, y))</td>
<td>(\forall x) i 6</td>
</tr>
<tr>
<td>7</td>
<td>(S(y_0, x_0))</td>
<td>Assumption</td>
</tr>
<tr>
<td>8</td>
<td>((\forall y) S(y_0, y))</td>
<td>(\forall y) i 7</td>
</tr>
<tr>
<td>9</td>
<td>((\forall x)(\forall y) S(x, y))</td>
<td>(\forall x) i 8</td>
</tr>
<tr>
<td>10</td>
<td>((\forall x)(\forall y) S(x, y))</td>
<td>e 3, 4-6, 7-9</td>
</tr>
<tr>
<td>11</td>
<td>((\forall x)(\forall y) S(x, y))</td>
<td>(\forall y) e 2, 3-10</td>
</tr>
<tr>
<td>12</td>
<td>((\forall x)(\forall y) S(x, y))</td>
<td>(\forall x) e 1, 2-11</td>
</tr>
</tbody>
</table>
Quantifier Proof Rules (for reference)
(\(\forall x\))-Elimination Rule (\(\forall x \ e\))

- \[
\frac{(\forall x \ e)}{(\forall x) \ [t/x]}
\]

where \(t\) is any term that is free for \(x\) in \(\forall x\).

- **What the rule says:**

If we have derived a universally-quantified formula \(\forall x\), then the formula \(\forall x\) with any (appropriately-qualified) **specific instance** of \(x\) substituted for \(x\) is derivable.
(□x)-Introduction Rule

- This rule uses a sub-derivation, with no formula assumed.

\[
\begin{array}{c}
\text{x}_0 \\
\vdots \\
\vdots \\
\vdots \\
\Box[x_0/x] \\
\hline
(\Box x)\Box
\end{array}
\]

- Here \(x_0\) is a “fresh” variable otherwise unused in the proof.
- \(x_0\) must be free for \(x\) in \(\Box\), but since \(x_0\) is “fresh”, this should never be an issue.
(∀x)-Introduction Rule

• **What this rule says:**

  • If we have argued to derive a term \( \square[x_0/x] \) where \( x_0 \) is an **arbitrary** value of \( x \), then we are justified in concluding \( (\forall x)\square \).

  • The key is the word “arbitrary”; there can be no constraints attached to \( x_0 \).

  • **Note:** Once the conclusion \( (\forall x)\square \) is drawn, \( x_0 \) is **discharged** and cannot be further used.
(\(\forall x\))-Introduction Rule (\(\forall x \ i\))

- \(\begin{array}{c}
\forall[t/x] \\
(\forall x) \ \Box
\end{array}\)

where t is any term that is free for x in \(\Box\).

- **What the rule says:**

If we have exhibited a formula \(\Box\) in which variable x is replaced by a **specific instance** then we can conclude that there is an x for which the formula is true.
(\(\forall x\))-Elimination Rule (\(\forall x \ e\))

\[
\begin{array}{c}
x_0 \\
\Box[x_0/x]
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\Box
\end{array}
\]

(\(\forall x\)) \[\Box\] (\(\forall x \ e\))

- Here \(x_0\) is a “fresh” variable otherwise unused in the proof.
- \(x_0\) must be free for \(x\) in \(\Box\), but since \(x_0\) is “fresh”, this should never be an issue.
(\(\forall x\))-Elimination Rule (\(\forall x \mathcal{E}\))

\[
\begin{array}{c}
v_0 \\
\vdash \forall[x_0/x] \\
\vdash \\
\vdash \mathcal{C}
\end{array}
\]

- **What this rule says:**
  - Assume that we have derived \((\forall x)\mathcal{E}\). One use we can make of this fact is to let \(x_0\) be an \(x\) such that \(\forall[x_0/x]\). There can be no other constraints on \(x_0\). If we then derive \(\mathcal{E}\) from the assumption about \(\mathcal{E}\), then we can conclude \(\mathcal{C}\) in general.