

# Natural Deduction for Predicate Calculus

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## Predicate Logic

- Proposition logic does not offer a way to talk about properties of **individuals**.
- This difficulty is overcome in predicate logic, which adds:
  - Constants representing individuals
  - Variables varying over individuals
  - Predicate symbols (including the equality symbol)
  - Function symbols
  - Quantifiers
- We use the same natural deduction framework, just augment the formula language and add new rules.
- “Truth” becomes more complex.

## Before: Propositional Language

- E is the start symbol
- E  $\rightarrow$  A |  $(\neg E)$  |  $(E \wedge E)$  |  $(E \vee E)$  |  $(E \rightarrow E)$  |  $\perp$  |  $\top$  // Atom  
// Negation (not)  
// Conjunction (and)  
// Disjunction (or)  
// Implication (implies)  
// Bottom  
// Top
- A  $\rightarrow$  'p' | 'q' | 'r' | 's' | ... // Propositions

## After: Predicate Language

- E is the start symbol
- E  $\rightarrow$  A |  $(\neg E)$  |  $(E \wedge E)$  |  $(E \vee E)$  |  $(E \rightarrow E)$  |  $\perp$  |  $\top$  |  $(\forall x)E$  |  $(\exists x)E$  // Atom (atomic formula)  
// Negation (not)  
// Conjunction (and)  
// Disjunction (or)  
// Implication (implies)  
// Bottom  
// Top  
// Universally-quantified formula  
// Existentially-quantified formula
- A now requires a more complex production

## Atomic Formulas

- $A \sqsubset P(L)$  // Predicate applied to list of terms
- $L \sqsubset T \mid T', L$  // List of terms
- $T \sqsubset V \mid C \mid F(L)$  // Term
  
- $V \sqsubset 'x' \mid 'y' \mid 'z' \mid \dots$  // Variable symbols
- $P \sqsubset 'p' \mid 'q' \mid 'r' \mid \dots$  // Predicate symbols
- $C \sqsubset 'a' \mid 'q' \mid 'c' \mid \dots$  // Constant symbols
- $F \sqsubset 'f' \mid 'g' \mid 'h' \mid \dots$  // Function symbols

Some predicates and functions may be abbreviated in infix form, e.g.  
= < < . . . will be infix predicate symbols  
+ \* / . . . will be infix function symbols  
We will not bother with a special grammar for these, although it can be done.

## Examples of Terms

- $b$  constant
- $y$  variable
- $g(b, y)$  function applications
- $g(h(b), c, h(y))$
- $g(a, b, g(a, b, c))$

## Examples of Atomic Formulas

- $p(b)$
- $q(y)$
- $p(g(b, y))$
- $r(a, g(h(b), c, h(y)))$

## Examples of Quantifier-Free Formulas

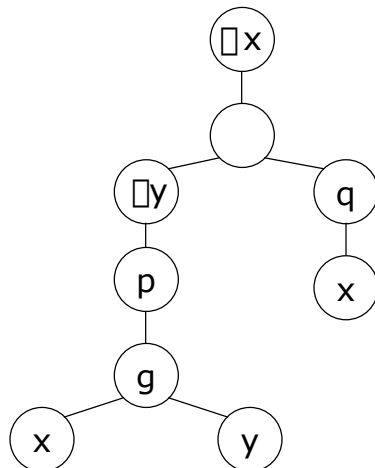
- $p(b) \wedge p(c)$
- $p(y) \vee q(y)$
- $p(g(b, y)) \vee q(y)$
- $\neg r(a, g(h(b), c, h(y)))$

## Examples of Formulas

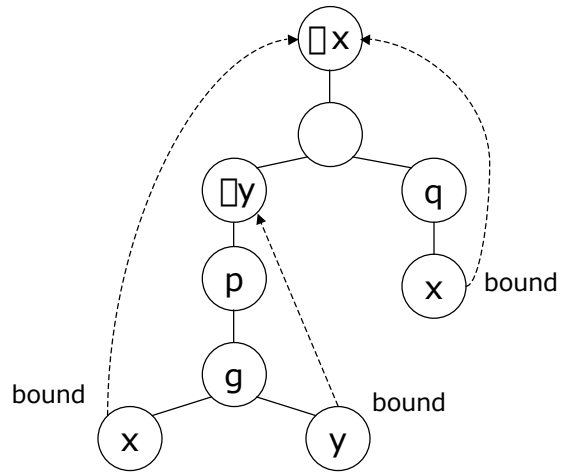
- $(\forall x)p(x)$
- $(\forall y) (p(y) \wedge q(y))$
- $(\forall y) (\forall x) (p(g(x, y)) \wedge q(y))$
- $(\forall x) ((\forall y) p(g(x, y))) \wedge q(x)$

## Syntax Trees (or "Parse" Trees)

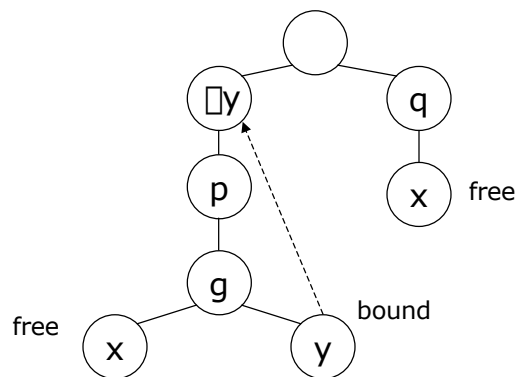
- We are assuming familiarity with syntax trees from CS 60.
- Here  $(\forall x)$   $(\exists x)$  are treated as 1-ary operators.
- Example:  $(\forall x) ((\forall y) p(g(x, y))) \wedge q(x)$



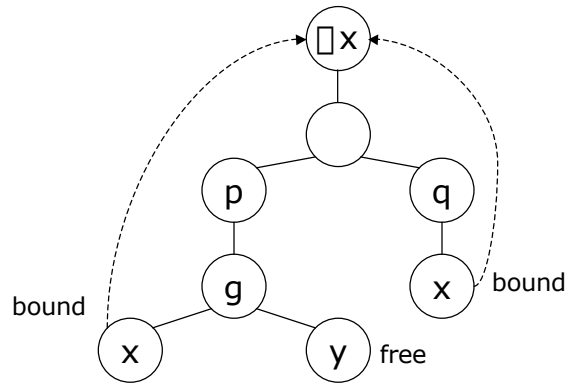
## Free and Bound Variable Instances



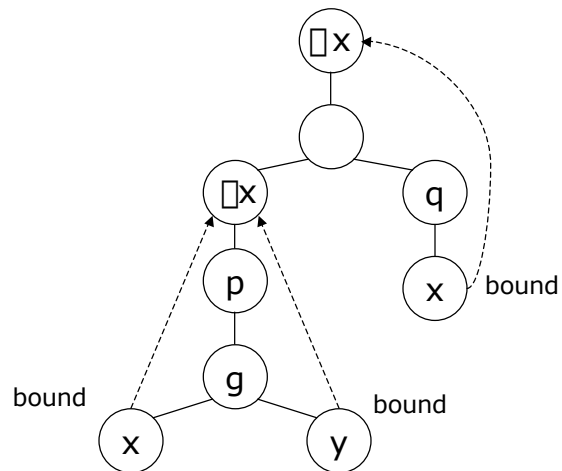
## Free and Bound Variable Instances



## Free and Bound Variable Instances



## Free and Bound Variable Instances



## Definition of Free and Bound

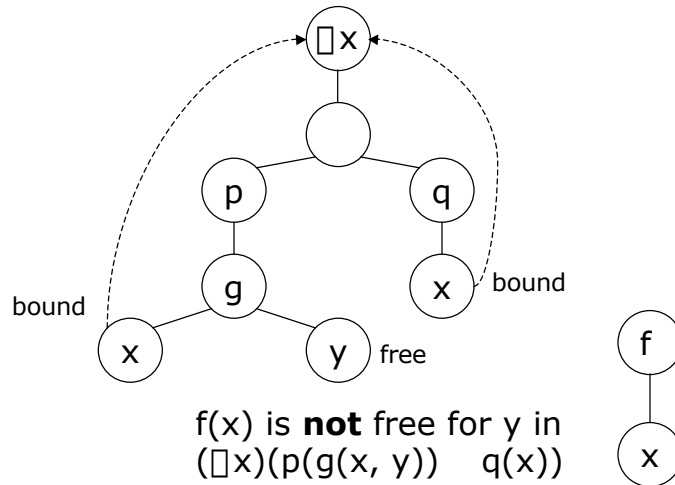
- In a term, every instance of a variable is free.
- If  $\phi$  is a formula, then any free instances of a variable  $x$  become bound in  $(\forall x)\phi$  and  $(\exists x)\phi$ .
- The free instances of variables in  $\phi$  and  $\psi$  remain free in  $(\phi\psi)$ ,  $(\phi \wedge \psi)$ ,  $(\phi \vee \psi)$ , and  $(\phi \rightarrow \psi)$ .
- The bound instances of variables in  $\phi$  and  $\psi$  remain bound in  $(\phi\psi)$ ,  $(\phi \wedge \psi)$ ,  $(\phi \vee \psi)$ , and  $(\phi \rightarrow \psi)$ .

## Substitutability

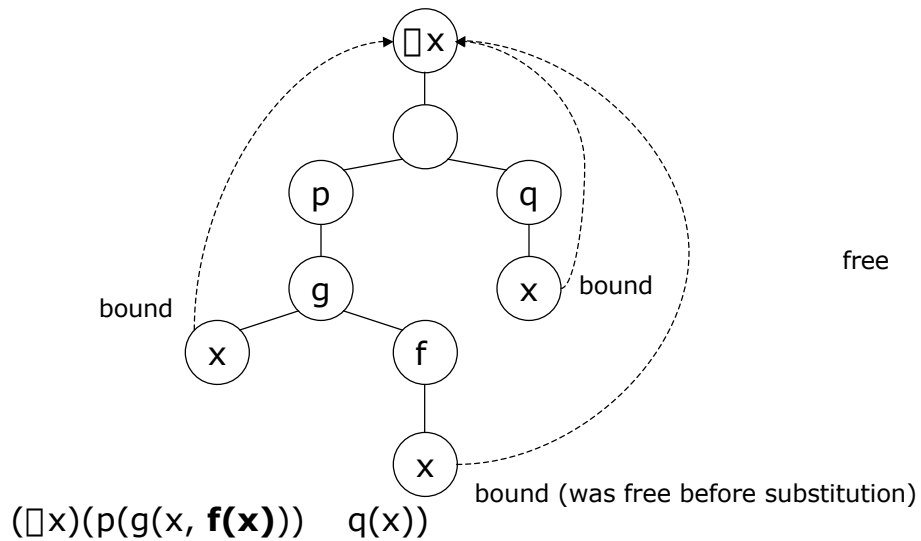
- We are going to need to be able to **substitute** terms for **free** variables in various formulas.
- While this is easy syntactically, there is a semantic restriction that must be observed:
  - In substituting a term for a variable within a formula, **no variables within the term can become bound** as a result of the substitution.
- If  $t$  is a term,  $v$  is a variable, and  $F$  is a formula, and the above restriction applies, we say that

**" $t$  is free for  $v$  in  $F$ ."**

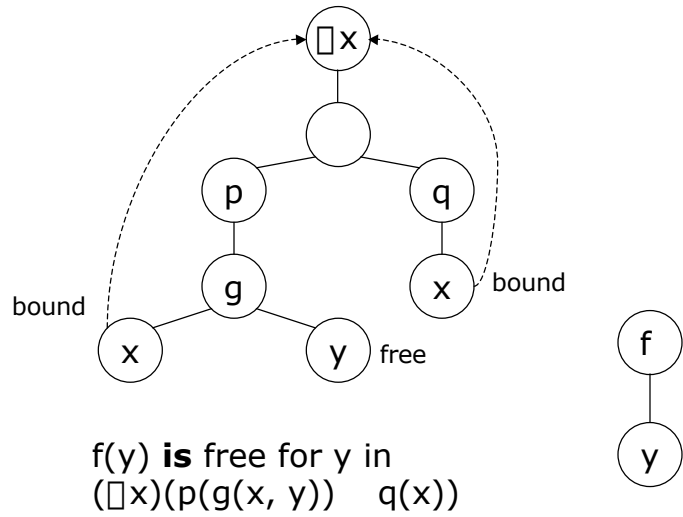
## Non-Substitutability Example



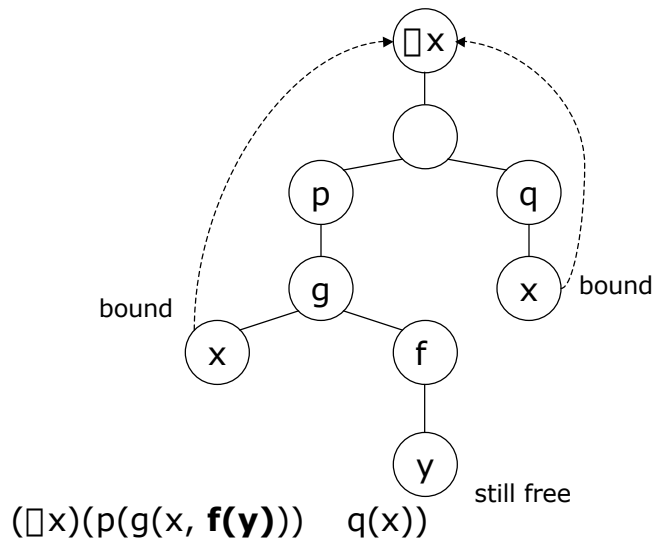
## Non-Substitutability Example



## Substitutability Example



## Substitutability Example



## Substitution Notation

- If  $t$  is a term,  $v$  is a variable, and  $F$  is a formula, and

$t$  is free for  $v$  in  $F$

then by

$F[t/v]$

we mean the result of substituting  $t$  for every **free** occurrence of  $v$  in  $F$ .

This notation is to be used **only** when the substitutability restriction applies.

Note:  $[ / ]$  is **meta**-syntax; these symbols do not appear in the resulting formula.

## Substitution Notation Example

Let  $F$  be the formula

$(\forall x)(p(g(x, y)) \rightarrow q(x))$

Let  $v$  be the variable  $y$ .

Let  $t$  be the term  $f(y)$ .

$f(y)$  **is** free for  $y$  in  $(\forall x)(p(g(x, y)) \rightarrow q(x))$ .

$F[f(y)/y]$  is  $(\forall x)(p(g(x, f(y))) \rightarrow q(x))$ .

## Substitution Notation Example

Let  $F$  be the formula

$$(\forall x)(p(g(x, y)) \rightarrow q(x))$$

Let  $v$  be the variable  $x$ .

Let  $t$  be the term  $f(y)$ .

$f(y)$  **is** free for  $x$  in  $(\forall x)(p(g(x, y)) \rightarrow q(x))$ ; there are no free instances of  $x$ .

$F[f(y)/y]$  is the same as  $F$ .

## Natural Deduction Rules

- We need introduction and elimination rules for each of:
  - $(\forall x)$
  - $(\exists x)$
  - $=$  (as a specially-interpreted predicate symbol)

## $(\forall x)$ -Elimination Rule ( $\forall x e$ )

- $$\frac{(\forall x) \phi}{\phi[t/x]} \quad (\forall x e)$$

where  $t$  is any term that is free for  $x$  in  $\phi$ .

- What the rule says:**

If we have derived a universally-quantified formula  $\phi$ , then the formula  $\phi$  with any (appropriately-qualified) **specific instance** of  $x$  substituted for  $x$  is derivable.

## Why the Qualification is Necessary

- $$\frac{(\forall x) \phi}{\phi[t/x]} \quad (\forall x e)$$

**where  $t$  is any term that is free for  $x$  in  $\phi$ .**

- Correct example:**  $z$  is free for  $x$  in  $(\forall y) p(y, x)$

1.	$(\forall x) (\forall y) p(y, x)$	Premise	
2.	$z = 0$	Assumption	
3.	$(\forall y) p(y, z)$	$(\forall x e) 1$	(substituting $z$ for $x$ )

- Incorrect example:**  $y$  is **not** free for  $x$  in  $(\forall y) p(y, x)$

1.	$(\forall x) (\forall y) p(y, x)$	Premise	
2.	$(\forall y) p(y, y)$	$(\forall x e) 1$	(substituting $y$ for $x$ )

- For instance,  $p$  could be  $>$  in the domain of natural numbers.



## $(\forall x e)$ $(\forall x i)$ Example

- Derive  $(\forall x)(p(x) \supset q(x))$ ,  $(\forall x) p(x) \supset (\forall x) q(x)$  :

1.	$(\forall x)(p(x) \supset q(x))$	Premise
2.	$(\forall x) p(x)$	Premise
3.	$x_0$ $p(x_0)$	$\forall x e$ 2
4.	$p(x_0) \supset q(x_0)$	$\forall x e$ 1
5.	$q(x_0)$	$\supset e$ 3, 4
6.	$(\forall x) p(x)$	$\forall x i$ 3-5

## $(\forall x e)$ $(\forall x i)$ English Equivalent

- Derive  $(\forall x)(p(x) \supset q(x))$ ,  $(\forall x) p(x) \supset (\forall x) q(x)$  :
- Assume  $(\forall x)(p(x) \supset q(x))$  and  $(\forall x) p(x)$ .

Let  $x_0$  be an arbitrary element.

From the second assumption, we have  $p(x_0)$ , and the first assumption  $p(x_0) \supset q(x_0)$ , hence also  $q(x_0)$  by *modus ponens*.

Since  $x_0$  was chosen arbitrarily, from  $q(x_0)$  we get  $(\forall x) q(x)$ .

## $(\exists x)$ -Introduction Rule ( $\exists x$ i)

- $$\frac{\phi[t/x] \quad (\exists x) \phi}{(\exists x) \phi}$$

where  $t$  is any term that is free for  $x$  in  $\phi$ .

- What the rule says:**

If we have exhibited a formula in which  $\phi$  variable  $x$  is replaced by a **specific instance** then we can conclude that there is an  $x$  for for which the formula is true.

## $(\exists x)$ -Introduction Rule ( $\exists x$ i)

- $$\frac{\phi[t/x] \quad (\exists x) \phi}{(\exists x) \phi}$$

where  $t$  is any term that is free for  $x$  in  $\phi$ .

- In essence, this rule loses information, by replacing knowledge of a **specific**  $x$  for which is true with the statement that there is some such  $x$ .
- It is analogous to rule  $\exists$ -Introduction.

## $(\exists x)$ -Elimination Rule ( $\exists x e$ )

$$\frac{(\exists x)\varphi \quad \begin{array}{l} x_0 \quad \varphi[x_0/x] \\ \vdots \\ \vdots \\ \varphi \end{array}}{\varphi} (\exists x e)$$

- Here  $x_0$  is a "fresh" variable otherwise unused in the proof.
- $x_0$  must be free for  $x$  in  $\varphi$ , but since  $x_0$  is "fresh", this should never be an issue

## $(\exists x)$ -Elimination Rule ( $\exists x e$ )

$$\frac{(\exists x)\varphi \quad \begin{array}{l} x_0 \quad \varphi[x_0/x] \\ \vdots \\ \vdots \\ \varphi \end{array}}{\varphi} (\exists x e)$$

- **What this rule says:**
- Assume that we have derived  $(\exists x)\varphi$ . One use we can make of this fact is to let  $x_0$  be **an**  $x$  such that  $\varphi[x_0/x]$ . There can be no other constraints on  $x_0$ . If we then derive  $\varphi$  from the assumption about  $\varphi$ , then we can conclude  $\varphi$  in general.

## $(\forall x \text{ i}) (\forall x \text{ e})$ Example

- Derive  $(\forall x)(p(x) \supset q(x))$ ,  $(\forall x) p(x) \vdash (\forall x) q(x)$ :

1.	$(\forall x)(p(x) \supset q(x))$	Premise
2.	$(\forall x) p(x)$	Premise
3.	$x_0$ $p(x_0)$	Assumption
4.	$p(x_0) \supset q(x_0)$	$\forall x \text{ e } 1$
5.	$q(x_0)$	$\supset \text{ e } 3, 4$
6.	$(\forall x) q(x)$	$\forall x \text{ i } 5$
7.	$(\forall x) q(x)$	$\forall x \text{ e } 3-6$

- In the  $\forall x \text{ e}$  rule,  $x$  is identified with  $p(x)$ , while  $\forall$  is identified with  $(\forall x) q(x)$ .
- Try not to be confused by the fact that  $\forall$  is in the conclusion.

## $(\forall x \text{ i}) (\forall x \text{ e})$ Example in English

- Derive  $(\forall x)(p(x) \supset q(x))$ ,  $(\forall x) p(x) \vdash (\forall x) q(x)$ :
- Assume  $(\forall x)(p(x) \supset q(x))$  and  $(\forall x) p(x)$ .

Let  $x_0$  be such that  $p(x_0)$ .

By the first assumption,  $p(x_0) \supset q(x_0)$ .  
Hence  $q(x_0)$ .

Since we've exhibited an  $x$  such that  $q(x)$ ,  
conclude  $(\forall x) q(x)$ .