Predicate Logic

• Proposition logic does not offer a way to talk about properties of individuals.
• This difficulty is overcome in predicate logic, which adds:
  • Constants representing individuals
  • Variables varying over individuals
  • Predicate symbols (including the equality symbol)
  • Function symbols
  • Quantifiers
• We use the same natural deduction framework, just augment the formula language and add new rules.
• “Truth” becomes more complex.
Before: Propositional Language

- E is the start symbol
- E ⊆ A
- (E ⊆ E) // Atom
- (E ⊆ E) // Negation (not)
- (E ⊆ E) // Conjunction (and)
- (E ⊆ E) // Disjunction (or)
- (E ' ⊆ ' E) // Implication (implies)
- ⊥ // Bottom
- T // Top

- A ⊆ ‘p’ | ‘q’ | ‘r’ | ‘s’ | … // Propositions
After: Predicate Language

- E is the start symbol
- E → A
  - (¬E)
  - (E ∨ E)
  - (E ∧ E)
  - (E → E)
  - (¬E)
  - (¬¬E)
  - T
  - (∀V)E
  - (∃V)E

- A now requires a more complex production
Atomic Formulas

- \( A \rightarrow P(L) \)  // Predicate applied to list of terms
- \( L \rightarrow T \mid T \,',', L \)  // List of terms
- \( T \rightarrow V \mid C \mid F(L) \)  // Term

- \( V \rightarrow 'x' \mid 'y' \mid 'z' \mid \ldots \)  // Variable symbols
- \( P \rightarrow 'p' \mid 'q' \mid 'r' \mid \ldots \)  // Predicate symbols
- \( C \rightarrow 'a' \mid 'q' \mid 'c' \mid \ldots \)  // Constant symbols
- \( F \rightarrow 'f' \mid 'g' \mid 'h' \mid \ldots \)  // Function symbols

Some predicates and functions may be abbreviated in infix form, e.g.
- \( = < < \ldots \) will be infix predicate symbols
- \( + \times / \ldots \) will be infix function symbols
We will not bother with a special grammar for these, although it can be done.
Examples of Terms

- $b$ constant
- $y$ variable
- $g(b, y)$ function applications
- $g(h(b), c, h(y))$
- $g(a, b, g(a, b, c))$
Examples of Atomic Formulas

- $p(b)$
- $q(y)$
- $p(g(b, y))$
- $r(a, g(h(b), c, h(y)))$
Examples of Quantifier-Free Formulas

- $p(b)$, $p(c)$
- $p(y) \lor q(y)$
- $p(g(b, y)) \Rightarrow q(y)$
- $\forall r(a, g(h(b), c, h(y)))$
Examples of Formulas

- $\forall x p(x)$
- $\forall y (p(y) \land q(y))$
- $\forall y (\forall x (p(g(x, y)) \land q(y))$
- $\forall x (\forall y p(g(x, y)) \land q(x))$
Syntax Trees (or “Parse” Trees)

- We are assuming familiarity with syntax trees from CS 60.
- Here (\(\cdot\)x) (\(\cdot\)x) are treated as 1-ary operators.
- Example: (\(\cdot\)x) ((\(\cdot\)y p(g(x, y)))) q(x))
Free and Bound Variable Instances

\[
x \vdash x
\]

\[
y \vdash y
\]

\[
p \vdash p
\]

\[
g \vdash g
\]

\[
x \vdash x
\]

\[
y \vdash y
\]
Free and Bound Variable Instances

Diagram:

- Free variables: \( x \) and \( y \)
- Bound variable: \( x \)
- Bound variable: \( y \)

Graph:

1. Two free variables connected by a dotted line: \( x \) and \( y \)
2. Bound variable \( y \) connected to free variable \( x \) by an arc.
3. Bound variable \( q \) connected to free variable \( x \) by a solid line.

Notations and Concepts:
- Free: \( x \) and \( y \)
- Bound: \( q \) and \( y \)
Free and Bound Variable Instances

\[ \Box x \]

\[ p \quad g \quad q \]

\[ x \quad y \]

bound

free
Free and Bound Variable Instances

Free and Bound Variable Instances
Definition of Free and Bound

- In a term, every instance of a variable is free.

- If $\phi$ is a formula, then any free instances of a variable $x$ become bound in $(\exists x)\phi$ and $(\forall x)\phi$.

- The free instances of variables in $\phi$ and $\psi$ remain free in $(\phi \land \psi)$, $(\phi \lor \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.

- The bound instances of variables in $\phi$ and $\psi$ remain bound in $(\phi \land \psi)$, $(\phi \lor \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.
Substitutability

• We are going to need to be able to substitute terms for free variables in various formulas.

• While this is easy syntactically, there is a semantic restriction that must be observed:
  
  • In substituting a term for a variable within a formula, no variables within the term can become bound as a result of the substitution.

• If t is a term, v is a variable, and F is a formula, and the above restriction applies, we say that

  “t is free for v in F.”
Non-Substitutability Example

f(x) is not free for y in 
(\[\forall x\](p(g(x, y)) \land q(x)))
Non-Substitutability Example

\[ (\Box x)(p(g(x, f(x))) \quad q(x)) \]

-bound (was free before substitution)
Substitutability Example

\[ f(y) \text{ is free for } y \text{ in } (\forall x)(p(g(x, y)) \quad q(x)) \]
Substitutability Example

\((\forall x)(p(g(x, f(y)))) \quad q(x))\)
Substitution Notation

- If $t$ is a term, $v$ is a variable, and $F$ is a formula, and $t$ is free for $v$ in $F$

  then by

  $F[t/v]$

  we mean the result of substituting $t$ for every free occurrence of $v$ in $F$.

This notation is to be used only when the substitutability restriction applies.

Note: $[ / ]$ is meta-syntax; these symbols do not appear in the resulting formula.
Substitution Notation Example

Let F be the formula

\[(\forall x)(p(g(x, y)) \land q(x))\]

Let v be the variable y.

Let t be the term f(y).

f(y) is free for y in \((\forall x)(p(g(x, y)) \land q(x))\).

F[f(y)/y] is \((\forall x)(p(g(x, f(y))) \land q(x))\).
Substitution Notation Example

Let $F$ be the formula

$$(\forall x)(p(g(x, y)) \land q(x))$$

Let $v$ be the variable $x$.

Let $t$ be the term $f(y)$.

$f(y)$ **is** free for $x$ in $(\forall x)(p(g(x, y)) \land q(x))$; there are no free instances of $x$.

$F[f(y)/y]$ is the same as $F$. 
Natural Deduction Rules

- We need introduction and elimination rules for each of:
  - \( \exists x \)
  - \( \forall x \)
  - \( = \) (as a specially-interpreted predicate symbol)
(\(\forall x\))-Elimination Rule (\(\forall x \ e\))

- \[
\frac{(\forall x) \emptyset}{\emptyset[t/x]}
\]

where \(t\) is any term that is free for \(x\) in \(\emptyset\).

- **What the rule says:**

If we have derived a universally-quantified formula \(\emptyset\), then the formula \(\emptyset\) with any (appropriately-qualified) **specific instance** of \(x\) substituted for \(x\) is derivable.
Why the Qualification is Necessary

• \(((\exists x)\) e)\[t/x]\]

where \(t\) is any term that is free for \(x\) in \(\exists x\).

- Correct example: \(z\) is free for \(x\) in \((\exists y)\; p(y, x)\)
  
  1. \((\exists x)\; (\exists y)\; p(y, x)\)  
     Premise
  
  2. \(z = 0\)  
     Assumption
  
  3. \((\exists y)\; p(y, z)\)  
     \((\exists x)\; e)\[z/x]\)  
     (substituting \(z\) for \(x\))

- Incorrect example: \(y\) is not free for \(x\) in \((\exists y)\; p(y, x)\)
  
  1. \((\exists x)\; (\exists y)\; p(y, x)\)  
     Premise
  
  2. \((\exists y)\; p(y, y)\)  
     \((\exists x)\; e)\[y/x]\)  
     (substituting \(y\) for \(x\))

- For instance, \(p\) could be > in the domain of natural numbers.
(□x)-Introduction Rule

• This rule uses a sub-derivation, with no formula assumed.

\[
\begin{array}{c}
\ x_0 \\
\vdots \\
\vdots \\
\vdots \\
\square[x_0/x] \\
\hline
(\square x) \square \\
\end{array}
\]

• Here \(x_0\) is a “fresh” variable otherwise unused in the proof.
• \(x_0\) must be free for \(x\) in \(\square\), but since \(x_0\) is “fresh”, this should never be an issue
(\(\exists x\))-Introduction Rule

- What this rule says:

- If we have argued to derive a term \(\exists [x_0/x]\) where \(x_0\) is an arbitrary value of \(x\), then we are justified in concluding \((\exists x)\)\(\exists\).

- The key is the word “arbitrary”; there can be no constraints attached to \(x_0\).

- Note: Once the conclusion \((\exists x)\)\(\exists\) is drawn, \(x_0\) is discharged and cannot be further used.
Example

Derive \( (\exists x) (\forall x) (p(x) \rightarrow q(x)), \ (\forall x) p(x) \vdash (\exists x) q(x) \):

1. \( (\exists x) (\forall x) (p(x) \rightarrow q(x)) \) Premise
2. \( (\forall x) p(x) \) Premise
3. \( x_0, p(x_0) \) \( \exists x e 2 \)
4. \( p(x_0) \rightarrow q(x_0) \) \( \exists x e 1 \)
5. \( q(x_0) \) \( \exists e 3, 4 \)
6. \( (\forall x) p(x) \) \( \forall x i 3-5 \)
(\(\exists x \ e\) \(\exists x \ i\)) English Equivalent

- Derive \((\exists x)(p(x) \land q(x)), (\exists x) p(x) \lor (\exists x) q(x)\):

- Assume \((\exists x)(p(x) \land q(x))\) and \((\exists x) p(x)\).

Let \(x_0\) be an arbitrary element.

From the second assumption, we have \(p(x_0)\), and the first assumption \(p(x_0) \land q(x_0)\), hence also \(q(x_0)\) by modus ponens.

Since \(x_0\) was chosen arbitrarily, from \(q(x_0)\) we get \((\exists x) q(x)\).
(\(\forall x\))-Introduction Rule (\(\forall x \, i\))

- \(\forall[i/t] \quad (\forall x \, i) \quad (\forall x) \quad \bot\)

where \(t\) is any term that is free for \(x\) in \(\bot\).

- **What the rule says:**

  If we have exhibited a formula in which \(\bot\) variable \(x\) is replaced by a **specific instance** then we can conclude that there is an \(x\) for which the formula is true.
(\(\forall x\))-Introduction Rule (\(\forall x \ i\))

- \(\frac{\square[t/x]}{(\forall x) \square}\)  (\(\forall x \ i\))

where \(t\) is any term that is free for \(x\) in \(\square\).

- In essence, this rule loses information, by replacing knowledge of a **specific** \(x\) for which is true with the statement that there is some such \(x\).
- It is analogous to rule \(\exists\)-Introduction.
(\(\forall x\))-Elimination Rule (\(\forall x \, e\))

Here \(x_0\) is a “fresh” variable otherwise unused in the proof.

\(x_0\) must be free for \(x\) in \(\square\), but since \(x_0\) is “fresh”, this should never be an issue.
(\(\forall x\))-Elimination Rule (\(\forall x\ e\))

\[
\begin{array}{c}
  x_0 \\
  \square [x_0/x] \\
  \quad . \\
  \quad . \\
  \quad . \\
  \quad . \\
  \quad . \\
\end{array}
\]

\[
(\forall x)\square \quad (\forall x\ e)
\]

- **What this rule says:**
  - Assume that we have derived (\(\forall x\)\(\square\)). One use we can make of this fact is to let \(x_0\) be an \(x\) such that \(\square [x_0/x]\). There can be no other constraints on \(x_0\). If we then derive \(\square\) from the assumption about \(\square\), then we can conclude \(\square\) in general.
(\exists x i) (\exists x e) Example

- Derive (\exists x)(p(x) \lor q(x)), (\exists x) p(x) \Rightarrow (\exists x) q(x):

  1. (\exists x)(p(x) \lor q(x)) \quad \text{Premise}
  2. (\exists x) p(x) \quad \text{Premise}
  3. x_0 \quad \text{Assumption}
     \begin{array}{l}
     \text{p}(x_0) \\
     \text{p}(x_0) \lor \text{q}(x_0) \\
     \text{q}(x_0) \\
     (\exists x) \text{q}(x) \\
     \end{array} \quad (\exists x) e 1, 3-4
  4. \quad (\exists x) q(x) \quad (\exists x) i 5
  5. \quad (\exists x) q(x) \quad \Rightarrow (\exists x) e 3-6

- In the (\exists x e) rule, \exists is identified with p(x), while \exists is identified with (\exists x) q(x).
- Try not to be confused by the fact that \exists is in the conclusion.
(∀x i) (∃x e) Example in English

- Derive (∀x)(p(x) ∨ q(x)), (∃x) p(x) \models (∃x) q(x):
- Assume (∀x)(p(x) ∨ q(x)) and (∃x) p(x).

Let x₀ be such that p(x₀).

By the first assumption, p(x₀) ∨ q(x₀).
Hence q(x₀).

Since we’ve exhibited an x such that q(x),
conclude (∃x) q(x).