Primitive and Partial Recursive Functions

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What is this?

- An alternate approach to defining the computable functions, one based on composition of numeric functions.
- Sometimes having this alternate viewpoint will be helpful.
- The family of primitive recursive functions is first defined, then partial recursive functions are built on that.

Primitive Recursive Functions

- The set of primitive recursive functions is given inductively.
- Every function is k-ary, for some $k \geq 0$.
- The domain and co-domain of each function is the set of natural numbers {0, 1, 2, 3, …} or k-tuples thereof.

Basis Functions (1 of 3)

- The **zero** functions are all primitive recursive:

$$Z^k(x_1, x_2, \ldots x_k) = 0$$

for each arity $k \geq 0$.

Basis Functions (2 of 3)

- The **projection** functions are all primitive recursive:

$$P^k_j(x_1, x_2, \ldots x_k) = x_j$$

for each arity $k \geq 0$ and each i, $1 \leq i \leq k$.

Basis Functions (3 of 3)

- The **successor** function is primitive recursive:

$$S(x) = x + 1$$

Induction Rules (1 of 2)

- The **composition** of primitive recursive functions is primitive recursive:

$$h(x_1, x_2, \ldots x_k) =$$

$$\begin{aligned} f(&g_1(x_1, x_2, \ldots x_k), \\ &g_2(x_1, x_2, \ldots x_k), \\ &\ldots \\ &g_r(x_1, x_2, \ldots x_k)) \end{aligned}$$

for each pair of arities k, r $\geq$ 0.

Constant Functions

- A consequence of the rules up to this point is that **constant** functions are all primitive recursive:

  $$C^k_c(x_1, x_2, . . . x_k) = c$$

  for each natural number c.

  This is so because is just a composition of the zero and successor functions:

  $C^k_c(x_1, x_2, . . . x_k) = S(S(\ldots S(zero(x_1, x_2, . . . x_k)) \ldots))$

Explicit Definition (ED)

- This is a convenience in lieu of showing stacks of compositions, projections, and constants, we can just use definitions such as:

  $$f(x, y, z) = g(h(y, x), 5, k(z, z))$$

  and know that if g, h, and k are primitive recursive, so is f.

Induction Rules (2 of 2)

- A function defined from primitive recursive functions by the following **primitive recursion pattern** is primitive recursive :

$$h(0, x_1, x_2, . . . x_k) = g(x_1, x_2, . . . x_k)$$

$$h(n+1, x_1, x_2, . . . x_k) =$$

$$f(x_1, x_2, . . . x_k, n, h(n, x_1, x_2, . . . x_k))$$

- Here h is being defined from g and h, which are known to be primitive recursive.

Examples of Primitive Recursive Functions

• add(x, y):	addition
• mult(x, y):	multiplication
• pred(x):	predecessor
• sub(x, y):	proper subtraction
• mod(x, y):	modulus
• div(x, y):	integer division (quotient)
• sqrt(x):	integer square root

rex implementations

- I will demonstrate some of these using rex rules. This allows the definitions to be tested readily.

- rex does not enforce the primitive recursive formalism, so we have to be careful not to “cheat”.

add implementation

- S(n) = n + 1;   // pretend this is built in
- add(0, y) => y;
- add(n+1, y) => S(add(n, y));
- For reference

  $h(0, x_1, x_2, \ldots x_k) = g(x_1, x_2, \ldots x_k)$

  $h(n+1, x_1, x_2, \ldots x_k) =$

  $f(x_1, x_2, \ldots x_k, n, h(n, x_1, x_2, \ldots x_k))$

mult implementation

- mult(0, y) => 0;
- mult(n+1, y) => add(y, mult(n, y));
- For reference

$$h(0, x_1, x_2, \ldots x_k) = g(x_1, x_2, \ldots x_k)$$

$$h(n+1, x_1, x_2, \ldots x_k) =$$

$$f(x_1, x_2, \ldots x_k, n, h(n, x_1, x_2, \ldots x_k))$$

pred implementation

- pred(0, y) =>
- pred(n+1, y) =>
- For reference

$$h(0, x_1, x_2, \ldots x_k) = g(x_1, x_2, \ldots x_k)$$

$$h(n+1, x_1, x_2, \ldots x_k) =$$

$$f(x_1, x_2, \ldots x_k, n, h(n, x_1, x_2, \ldots x_k))$$

sub implementation

- sub is **proper** subtraction (aka "monus"): If a < b, then sub(a, b) = 0.

- sub(y, 0) =>

- sub(y, n+1) =>

- For reference

$$h(0, x_1, x_2, \ldots x_k) = g(x_1, x_2, \ldots x_k)$$

$$h(n+1, x_1, x_2, \ldots x_k) =$$

$$f(x_1, x_2, \ldots x_k, n, h(n, x_1, x_2, \ldots x_k))$$

Primitive Recursive Predicates

- For some definitions we want to have predicates, which we can equate to functions return only values 0 (false) and 1 true.
- sgn(0) => 0;
- sgn(n+1) => 1;
- sgn converts arbitrary values to {0, 1}.

Negation

- not(0) => 1;
- not(n+1) => 0;

Equality Predicate

- eq(x, y) = not(add(sub(x, y), sub(y, x)));

if-then-else function

- ifthenelse(0, x, y) => y;
- ifthenelse(n+1, x, y) => x;

- This can be inefficient if all arguments must be computed to get a result (applicative order). It should be taken as an “academic” version, perhaps.

mod and div

```
mod(0, y) => 0;

mod(n+1, y) => ifthenelse(eq(S(mod(n, y)), y),
                          0,
                          S(mod(n, y)));

div(0, y) => 0;

div(n+1, y) => ifthenelse(eq(S(mod(n, y)), y),
                          S(div(n, y)),
                          div(n, y));
```

Perspective

- Primitive recursive functions are functions that can be defined using only **definite iteration** (e.g. for-loop with upper bound pre-determined)

  and not requiring indefinite iteration (while-loops) or the full power of recursion.

- Primitive recursion as given is not a special case of tail recursion, although there is an equivalent version that is.

Primitive Recursion = Definite Iteration

- The function h defined in the primitive recursion scheme can be computed by the following for-loop:

```
acc = g(x_1, x_2, . . . x_k);
for( j = 0; j < n; j++ )
        acc = f(x_1, x_2, . . . x_k, j, acc);

// acc == h(n, x_1, x_2, . . . x_k)
```

Tail-Recursive Version

- The function $h(n, x_1, x_2, . . . x_k)$ can be computed as $t(n, g(x_1, x_2, . . . x_k))$ where t is defined in the following tail-recursion:

$$t(0, acc) = acc;$$

$$t(n+1, acc) = f(x_1, x_2, . . . x_k, n, acc);$$

Example: Factorial

- Primitive-recursive version
  (uses the primitive-recursion pattern):

  fac(0) = 1
  fac(n+1) = mult(n+1, fac(n))

- Tail-recursive version
  (doesn’t use the pattern, but equivalent):

  fac_tr(n) = t(n, 1)

  t(0, acc) = acc
  t(n+1, acc) = t(n, mult(n+1, acc))

Tail-Recursion Theorem

- If h is defined from f and g using primitive recursion, then h can also be defined from f and g using tail recursion.
- The conversion is given on preceding slides.
- Claim: (□n) t(n, acc) = h(n, x_1, x_2, . . . x_k) provided acc is initialized with g(x_1, x_2, . . . x_k)).
- Proof is by induction on n.
  Show that both

  t(n, acc) = f(X, n-1, f(X, n-2, …, f(X, 0, g(X))…))
  h(n, X) = f(X, n-1, f(X, n-2, …, f(X, 0, g(X))…))

Totality Theorem

- Every primitive recursive function is total.
- Two levels of induction are involved:
  - For each use of the primitive-recursion pattern, there is an induction to show that h is defined for all n, assuming that f and g are total.
  - There is induction on the number of uses of the induction rules in defining the top level function.

Computability Theorem

- Every primitive-recursive function is computable by a Turing machine.
- This is more-or-less obvious, but it can be shown in significant detail by showing how a Turing machine can be constructed by composing functions using the basis functions and induction rules.
- Please consult the text for details.

Diagonalization Theorem

- There is a total recursive function that is not primitive recursive.
- Proof: Using the computability theorem, we know that we can enumerate the primitive recursive functions of one argument:

  $p_0, p_1, p_2, \ldots$

  They are just a subsequence of the ones computed by Turing machines in the ordering of all Turing machines.

  Then define $q(x) = p_x(x) + 1$. This function is clearly total, since each p_x is, but q cannot appear in the list.

The Ackermann Hierarchy

- We notice that add and mult have similar definitions.
  - add uses S as a base
  - mult uses add as a base
- We can go on to define exp analogously, using mult as a base.
- When does this stop?
- We quickly reach functions that have very large values for small arguments.
- It is possible to diagonalize over this hierarchy.

The Ackermann Hierarchy

- $A_0(m) = S(m)$
- $A_{n+1}(0) = A_n(1)$
- $A_{n+1}(m+1) = A_n(A_{n+1}(m))$

- Each function in the list: A_0, A_1, A_2, . . . is clearly primitive-recursive.
- Define $A(n, m) = A_n(m)$.
- It can be proved that A **grows faster** than any single primitive-recursive function, hence is not primitive-recursive itself.

Recursive Functions

- This is a family of partial functions, also known as “**the** partial recursive functions”.
- We have used that term to describe the partial computable functions, and the definitions turn out to be equivalent.

μ-Recursive Functions

- Start with the primitive-recursive functions as a base.
- Add one more induction rule: If f is a k+1 ary μ–recursive function, h is a a k+1 ary one:

$h(x_1, x_2, . . . x_{k-1}) =$

$$\mu x_k\ [f(x_1, x_2, . . . x_k) = 0]$$

read "the least value of x_k such that $f(x_1, x_2, . . . x_k) = 0$.

□-Recursive Functions

- The definition of a function using the operator really only makes sense if the function f is **total**.
- Totality cannot be established syntactically (why?).
- We will adopt the convention that the values of for which f is computed are given sequentially, and that if $f(x_1, x_2, \ldots x_k)$ is divergent for any value before reaching a value x_k such that $f(x_1, x_2, \ldots x_k) = 0$, then $h(x_1, x_2, \ldots, x_{k-1})$ also diverges for the given arguments.

Computability Theorem for μ-Recursive Functions

- We can extend to the partial recursive functions the proof that the primitive recursive functions are Turing computable.
- Consult the text for details.

Converse of the Computability Theorem

- Every Turing computable partial function is computable by a μ-recursive partial function.

- Moreover, the μ operator needs to be used only **once** to achieve any partial-recursive function.

Importance of the Computability Therem and its Converse

- Turing-computable partial functions and μ-recursive partial functions are established as being the same thing.

- One was defined using **strings**, the other using **numbers**.

Establishing the Converse

- The converse shows that any Turing-computable partial function is a □-recursive partial function.
- To do this involves **encoding** TM tapes and configurations as numbers.
- Then it can be shown that there are primitive recursive functions that:
  - Simulate a single step of a Turing machine.
  - Tell whether an encoded configuration is halting.

Primitive Recursive TM equivalents

- R(x) is the configuration resulting after 1 step from x.
- T(i, x) is the configuration resulting from configuration x after i steps.
- P(x) indicates whether or not a configuration is halting (0 or 1).

Recursive TM equivalents

- Halting in i steps is expressed by:

$$\square i \, [P(T(i, x_0)) = 0]$$

- The halting configuration is:

$$T(\square i \, [P(T(i, x_0)) = 0], x_0)$$

Encodings

- Using **primitive** recursive functions to encode and decode tapes and configurations requires a lengthy, but interesting, excursion.

- One way (but not the only way) to encode arbitrary sequences of numbers is to use "Gödel numbering":
  Any sequence of natural numbers
  $(x_1, x_2, \ldots, x_k)$
  can be encoded as a **single** natural number:
  $p_1^{x1} p_2^{x2} \ldots p_k^{xk}$

Universal □-Recursive Functions

- Most results for Turing machines have parallels for the □-Recursive Functions
- The □-recursive functions are programs that can be coded and enumerated just like Turing machines can:

  $□^k_0, □^k_1, □^k_2, □^k_3, \ldots$

  are the k-ary □-recursive functions for any fixed k.

Kleene's Normal Form Theorem

- For each $k \geq 1$, there exists a 1-ary primitive recursive function U and a (k+2)-ary primitive recursive predicate T_k such that
  - $\square^k_n(x_1, x_2, . . . , x_k)\square$ iff $(\square z)$ $T(n, x_1, x_2, . . . , x_k, z)$

    $\square^k_n(x_1, x_2, . . . , x_k) = U(\square z\ [T(n, x_1, x_2, . . . , x_k, z) = 0])$
- Essentially, T is like the function that tells whether the n^{th} configuration of a TM computation is halting, while U gives the result from that halting configuration.
- The numbers z code both the program **and** the number of steps.

Universal □-Recursive Functions

- For each k, there is a □-recursive function □ of k+1 variables such that

$$□(n, x_1, x_2, . . . , x_k) =$$

$$□^k_n(x_1, x_2, . . . , x_k)$$

- □ is a universal function for k arguments.

Recursive and R.E. Sets

- Languages are now sets of numbers.
- A set is recursive if its characteristic function is total recursive.
- A set is recursively-enumerable if there is a total recursive function that enumerates it.
- Equivalently, a set is recursively-enumerable if it is the domain of some partial recursive function.

Halting and Divergence

- $\varphi(x)\downarrow$ is used to mean that φ has a value for argument x.
- $\varphi(x)\uparrow$ is used to mean that φ diverges on argument x.
- The set $\{j \mid \varphi_j(j)\uparrow\}$ is not recursively-enumerable: this is the **halting problem**.
- The set $\{j \mid \varphi_j(j)\downarrow\}$ is recursively-enumerable, but not recursive.
- The set $\{j \mid \varphi_j(0)\uparrow\}$ is not R.E. either, analogous to the blank-tape halting problem.

Problem: Exhibit a total function that is not computable.

- Define

$$f(j) = \begin{cases} \square_j(j) + 1 \text{ if } \square_j(j)\square \\ 0 \text{ otherwise} \end{cases}$$

f is evidently total. f is not computable, since if it were, say $\square_k$, then f(k) would give a contradiction.

Hilbert's Tenth Problem

- This is an unsolvable problem of practical interest.
  - Give an algorithm that will determine whether a multi-variate polynomial equation has an **integer** solution.
  - Example of an equation:
    $x^2 + 3y^3 + 13 = 0$
- The problem of whether this is possible was posed by Hilbert in 1900.
- This problem was not proved unsolvable until 1970, when it was established that **every** recursively-enumerable set of k-tuples is the set of solutions of some such equation.