Soundness and Completeness of Propositional Natural Deduction

Robert Keller

23 November 2003

Validity vs. Provability

- $\square_1, \ldots, \square_n$ |□ □ means □ is **provable** from $\square_1, \ldots, \square_n$
- $\square_1, \ldots, \square_n$ |= □ means roughly the following:

  If each of $\square_i$ is true, then □ is true.
- In other words, □ is a **valid** conclusion from $\square_1, \ldots, \square_n$.
- We need a definition of ***truth*** to make this precise.

Soundness vs. Completness

- **Soundness**: Every provable sequent is valid:

  (for every $□_1, \ldots, □_n, □$):

  $□_1, \ldots, □_n$ |□ □ implies $□_1, \ldots, □_n$ |= □

- **Completeness**: Every valid sequent is provable:

  (for every $□_1, \ldots, □_n, □$):

  $□_1, \ldots, □_n$ |= □ implies $□_1, \ldots, □_n$ |□ □

Definition of "Truth"

- A **valuation** is a mapping from proposition symbols {p, q, r, …} to the set {**T**, **F**}.
- A valuation □ is **extended** to an arbitrary formulas inductively as follows:
  - □(T) = **T**.
  - □(□) = **F**.
  - □(□□□) = **T** iff □(□) = **T** and □(□) = **T**.
  - □(□ □) = **T** iff □(□) = **T** or □(□) = **T**.
  - □(□□ □) = **T** iff □(□) = **F** or □(□) = **T**.
  - □(□□) = **T** iff □(□) = **F**.

Definition of Validity

- $\square_1, \ldots, \square_n$ |= $\square$ means

  For each valuation $\square$:

  If, for each i, $\square(\square_i)$ = **T**

  then also $\square(\square)$ = **T**.

- $\square_1, \ldots, \square_n$ |= $\square$ is sometimes read $\{\square_1, \ldots, \square_n\}$ ***entails*** $\square$.

- When |= $\square$ (i.e. the LHS of |= is empty), $\square$ is a ***tautology***, a concept with which we assume familiarity.

Proof of Soundness

- **Soundness**: Every provable sequent is valid:

  (for every $\square_1, \ldots, \square_n, \square$):

  $\square_1, \ldots, \square_n$ |$\square$ $\square$ implies $\square_1, \ldots, \square_n$ |= $\square$

- Assume that $\square_1, \ldots, \square_n$ |$\square$ $\square$, to show $\square_1, \ldots, \square_n$ |= $\square$.

- This will be by **strong induction** on the length of the proof of $\square$.

- Basis: If the length of the proof is 1, then $\square$ is one of the $\square_i$ so $\square_1, \ldots, \square_n$ |= $\square_i$ holds trivially.

Proof of Soundness Continued

- Induction Step: Suppose that $□_1, \ldots, □_n \vdash □$ implies $□_1, \ldots, □_n \models □$ for every proof that uses fewer than k steps ($k > 1$).
- We wish to show the same for a proof that uses k steps.
- Let □ be a valuation such that each i, $□(□_i)$ = **T**.
  We need to show that □(□) = **T** as well.
- Since □ is the last formula in the proof, it must be the consequent of one of the rules of inference, and the antecedents of that rule occurred earlier in the proof.
- Since the antecedents occurred earlier in the proof, their derivations required fewer than k steps, so by the induction hypothesis, □(□) = **T** for each antecedent □.
- So if we can show that whenever a valuation makes all antecedents true it also makes the consequent true, we are done.

Proof of Soundness Continued

- So if we can show that whenever a valuation makes all antecedents true it also makes the consequent true (which we will call **truth-preservation**), we are done.
- Now we seen one reaons to try to **keep the number of non-derived rules small**: it creates less work in this kind of a proof.
- The next slides show examples of truth-preservation for the various rules.

Truth Preservation for (□i)

- $\underline{\square\quad\square}$
  □□□

- The definition of the extended □ says that □(□□□) = **T** iff □(□) = **T and** □(□) = **T.**

Truth Preservation for (□e_1, □e_2)

- $\underline{\square\square\square}$ (□e_1)
  □

- $\underline{\square\square\square}$ (□e_2)
  □

- The definition of the extended □ says that □(□□□) = **T** iff □(□) = **T and** □(□) = **T.**

Truth Preservation for (i_1, i_2)

- $\dfrac{□}{□ \quad □}$ (i_1)

- $\dfrac{□}{□ \quad □}$ (i_2)

- The definition of the extended □ says that □(□   □) = **T** iff □(□) = **T or** □(□) = **T.**

- Whichever rule was used, we know that the valuation of the antecedent is **T**, therefore □(□   □) = **T.**

Truth Preservation for □ -Elimination Rule

- $\dfrac{□,\ □□\ □}{□}$ (□ e)

- We know that □(□□ □) = **T** iff □(□) = **F** or □(□) = **T**.

- By the induction hypothesis, since both □ and □□ □ occurred earlier in the proof, □(□) = **T** and □(□□ □) = **T.**

- Since □(□) ≠ **F** but □(□□ □) = **T**, it must be the case that □(□) = **T**.

The Rules Employing Sub-Proofs

- Here's where the going gets messy.
- Sub-proofs are in boxes.
- From the perspective of the outer proof, the boxes provide part of the justification for **single steps** that would not otherwise follow from previous steps before the box.
- We must argue that for each valuation that makes the premises **and the assumptions inside the box** true, the conclusion is true.
- Since such valuations "include" any valuation that makes the premises true, and since the derivation inside the box is length less than k, we are justified in saying that **the conclusion at the end of the box** must also be true.

From the Authors' errata:

(http://www.doc.ic.ac.uk/~mrh/lics/errata2.pdf)

- p. 57–60, the soundness proof: the formal notion of a "modified proof" is problematic and does not pass formal muster (courtesy of James Caldwell). Our apologies! At any rate, the inductive argument can still be appreciated with an intuitive understanding of "modified proofs". We mean to re-write the definition of "modified proof" in the second, upcoming edition.

The following is my version of the rest of the soundness proof.

Truth Preservation for □ i

□
.
.
.
□

□□ □ (□ i)

- For a valuation □ such that □(□) = **T**, as well as □(□) = **T** for every premise □, it follows from the induction hypothesis that □(□) = **T**. Therefore □(□□ □) = **T**.

Truth Preservation for e

- This rule uses two sub-derivations:

□ □ | □ . . □ | □ . . □ | (e)

□

- Similar to □ i, but now a valuation □ such that □(□ □) = **T** must make either □(□) = **T** or □(□) = **T**. Hence by the induction hypothesis, it will make □(□) = **T**.

Truth Preservation for □□e

- □□□ (□□e)
  □

- A valuation □ such that □(□□□) = **T** will have □(□□) = **F**, so □(□) = **T**.

Truth Preservation for □i

$$\frac{\boxed{\begin{array}{c}\square\\ \vdots\\ \square\end{array}}}{\square\square}\ (\square i)$$

If there were a valuation □ that makes □(□) = **T**, then by the induction hypothesis, □(□) = **T**, which is impossible. Therefore, there is no such valuation, hence □(□□) = **T**.

Truth Preservation for □e

$$\frac{\square \ \square \ \square\square}{\square} \qquad (\square e)$$

No valuation □ that makes □(□ □ □□) = **T**, so every such valuation (namely none) makes □(□) = **T** vacuously.

Truth Preservation for □e

$$\frac{\square}{\square} \qquad (\square e)$$

No valuation □ that makes □(□) = **T**, so every such valuation (namely none) makes □(□) = **T** vacuously, regardless of what □ is.

Other Soundness Proofs

- There is probably one in Gentzen's original work.
- There is a much terser proof in the book *Logic and Structure* by van Dalen.
- He also uses a smaller rule set (6 rules), by defining □ as □ □ and defining using □ and □. His rules are □i, □e, □ i, □ e, □e, and RAA.

Uses of Soundness

- There is an algorithm for determining whether or not

  $\square_1, \ldots, \square_n \models \square$

- Thus, one can compute a **necessary** condition of whether there is a proof of

  $\square_1, \ldots, \square_n$ |□ □

Completeness

- Completeness says (for all $□_1, \ldots, □_n, □$)

  $□_1, \ldots, □_n \models □$

  implies

  $□_1, \ldots, □_n |□ \; □$

- If this could be established, then the algorithm mentioned for soundness would be a necessary and **sufficient** condition for the existence of a proof. That is, provability would be **solvability**.

Proof of Completeness

Three steps are used:

1. $□_1, \ldots, □_n \models □$ implies $\models (□_1 \; □ \; (□_2 \; □ \; . \; . \; . \; (□_n \; □ \; □ \;)) \ldots)$
2. For any formula $□$, $\models □$ implies $|□ \; □$.
3. $|□ \; (□_1 \; □ \; (□_2 \; □ \; . \; . \; . \; (□_n \; □ \; □ \;)) \ldots)$ implies $□_1, \ldots, □_n \, |□ \; □$

   Step 2 is the key one, as only it bridges the gap between $\models$ and $|□$. The other two are simplifying steps.

Proof that $|= (\square_1 \square\ (\square_2 \square\ . . . (\square_n \square\ \square\ ))\ …)$

implies $|\square\ (\square_1 \square\ (\square_2 \square\ . . . (\square_n \square\ \square\ ))\ …)$

- Given a formula of the indicated form that is true for every valuation, we need to construct a proof.
- The proof that would be constructed by the uniform method that will be developed might not be the one that we'd give left on our own. It will generally be more complex than necessary.

Idea of the proof that $|= \square$ implies $|\square\ \square$

- Assume $|= \square$.
- Let p_1, p_2, . . ., p_k be the set of all proposition symbols that occur in $\square$.
- For each combination of proposition symbols with and without negation, show that there is a sequent with that combination on the left and the formula of interest on the right:
  - $p_1, \ p_2, . . ., p_k \ |\square\ \square$
  - $\square p_1, \ p_2, . . ., p_k \ |\square\ \square$
  - $p_1, \square p_2, . . ., p_k \ |\square\ \square$
  - $\square p_1, \square p_2, . . ., p_k \ |\square\ \square$ etc.
- Then those 2^k sequents will be combined into a single sequent of the required form.

The Combination Process

- Because this constructs a derivation that is of length exponential in k, we will show it by example, for k = 2.
- Given that we have:
  - p_1, p_2 |□ □
  - □p_1, p_2 |□ □
  - p_1, □p_2 |□ □
  - □p_1, □p_2 |□ □
- The proof constructed for the single sequent is shown on the next page.

Proof Constructed for the Single Sequent

1. p_1 □p_1 — LEM
2. p_1 — Assumption
3. p_2 □p_2 — LEM
4. p_2 — Assumption
   . . . steps in the proof of **p_1, p_2** |□ □
5. □
6. □p_2 — Assumption
   . . . steps in the proof of **p_1, □p_2** |□ □
7. □
8. □ — e 3, 4-5, 6-7
9. □p_1 — Assumption
10. p_2 □p_2 — LEM
11. p_2 — Assumption
    . . . steps in the proof of **□p_1, p_2** |□ □
12. □
13. □p_2 — Assumption
    . . . steps in the proof of **□p_1, □p_2** |□ □
14. □
15. □ — e 10, 11-12, 13-14
16. □ — e 1, 2-8, 9-15

Proofs for the Individual Sequents

- We are left with showing that each of the individual sequents
  - $p_1,\ \ p_2, \ldots, p_k \mid\square\ \ \square$
  - $\square p_1,\ \ p_2, \ldots, p_k \mid\square\ \ \square$
  - $p_1, \square p_2, \ldots, p_k \mid\square\ \ \square$
  - $\square p_1, \square p_2, \ldots, p_k \mid\square\ \ \square$ etc.

  has a proof, given that
  - $\mid= \square$.

Proofs for the Individual Sequents

- For any formula $\square$, we want to show that $\mid= \square$ implies each of the individual sequents below has a proof
  - $p_1,\ \ p_2, \ldots, p_k \mid\square\ \ \square$
  - $\square p_1,\ \ p_2, \ldots, p_k \mid\square\ \ \square$
  - $p_1, \square p_2, \ldots, p_k \mid\square\ \ \square$
  - $\square p_1, \square p_2, \ldots, p_k \mid\square\ \ \square$ etc.

  where $p_1, p_2, \ldots, p_k$ are the proposition symbols in $\square$.

- Consider any combination $p^*_1, p^*_2, \ldots, p^*_k$ of the symbols negated or un-negated (e.g. $\square p_1, p_2, \ldots, \square p_k$) and the corresponding **valuation** that makes $\square(p^*_1 \square p^* \square \ldots \square p^*_k) = \mathbf{T}$.
  - If $\square(\square) = \mathbf{T}$ then $p^*_1, p^*_2, \ldots, p^*_k \mid\square\ \ \square$.
  - If $\square(\square) = \mathbf{F}$ then $p^*_1, p^*_2, \ldots, p^*_k \mid\square\ \ (\square\square)$.

Proving 1. If □(□) = **T** then $p^*_1, p^*_2, \ldots, p^*_k$ |□ □.

2. If □(□) = **F** then $p^*_1, p^*_2, \ldots, p^*_k$ |□ (□□).

- This is done by **induction** on the **structure** of the formula □, i.e. as determined by the grammar for formulas.
- **Basis**: If □ is a single proposition p, then
  - If □(p) = **T**, then p* must be p, and we have p |□ p.
  - If □(p) = **F** , then p* must be □p, and we have □p |□ □p.
  - If □ is □, then □(□) = **F**, so we can't have |= □.
  - If □ is T, then □(T) = **T**, and we know that |□ T.
- Induction Step: We have to show that the inductive hypothesis implies the conclusion for each possible operator: □□ □ .

Case where □ is of form □□

- If □(□) = **T**, then □(□) = **F**. By the induction hypothesis, part 2:
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ (□□),
  therefore
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ □, which is statement 1.
- If □(□) = **F**, then □(□) = **T**. By the induction hypothesis, part 1:
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ □.

  Using □□i to extend the proof one step, we have
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ □(□□).
  Therefore
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ □□, which is statement 2.

Case where □ is of form $□_1$ □ $□_2$

- We need to consider 4 cases: □($□_1$ $□_2$) = **FF**, **FT**, **TF**, **TT**.
- FF: In this case □($□_1$ □ $□_2$) = **F**.
  By the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ (□$□_1$)
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ (□$□_2$).
  Using □i, we get a proof of
  $p^*_1, p^*_2, \ldots, p^*_k$ |□ □($□_1$□ $□_2$)
  as desired.
- **The other 3 cases for $□_1$ □ $□_2$ are similar.**
- **The cases for the other operators (, □) are similar.**
- This concludes our sketch of the proof of the completeness theorem.