The material that we are covering in class this week is addressed in our textbook in Chapter 16, sections 16.1 and 16.2 and Chapter 17, sections 17.1 and 17.2.

1. **[25 Points] Hurts Car Rental!** Hurts Car Rental is experimenting with a new type of vehicle. These vehicles use a modular fuel pack which allows the vehicle to travel exactly 100 miles. (The fuel is believed to consist of a mixture of pulverized Spam, Mountain Dew, and Cheetos, although the technology is proprietary and this is just speculation.) When the car needs to be refueled, the fuel pack is removed from the car and is replaced by a new one. Thus, it is possible that a fuel pack will be replaced before it is completely used up. Moreover, the driver gets no credit for the remaining fuel in the fuel pack and can only purchase a new fully charged 100 mile pack.

A driver may wish to travel on a long freeway from one point to another. Since only designated service stations sell the fuel packs, the driver needs to plan carefully where to refuel. Moreover, each service station charges a different amount for a fuel pack. Some drivers also don’t want to stop very often to refuel. Consequently, these vehicles have a dial on the dashboard called the $\alpha$ dial. By turning the dial, the driver sets a value of $\alpha$ between 0 and 1. By doing so, the driver stipulates that she wishes to minimize the quantity $\alpha S + (1 - \alpha)C$ where $S$ is the number of fuel stops made and $C$ is the total cost paid for the fuel packs. Notice that when $\alpha$ is set to 0, the objective becomes that of minimizing the total cost paid for the fuel packs. When $\alpha$ is set to 1, the objective becomes that of minimizing the number of stops. When $\alpha$ is set somewhere between 0 and 1, the objective is a linear combination of these.

Consider a highway represented by a line segment. Let $p_1, p_2, \ldots, p_n$ be $n$ points on the line segment sorted from left to right where $p_1$ is the starting point, $p_n$ is the destination point, and each of these points has a service station selling fuel packs. Let $d_i$ be the distance from point $p_1$ to point $p_i$, for $1 \leq i \leq n$. Let $c_i$ be the cost of a fuel pack at the station at point $p_i$.

(a) Professor I. Lai claims that the following greedy algorithm minimizes the total cost when $\alpha = 0$: Buy a fuel pack at $p_1$ (we have no choice there - we need fuel to depart on the trip). Travel to the furthest point which is at most 100 miles away. Purchase a fuel pack there. Now repeat this process of traveling as far as possible on the current fuel pack before purchasing a new fuel pack. Explain briefly why this approach does not guarantee an optimal solution when $\alpha = 0$.

(b) Assume that a given value of $\alpha$ has been established by the driver. Give pseudocode for a recursive algorithm called `optimize([p_1, p_2, p_3, \ldots, p_j], p_k)` which returns the minimum value of $\alpha S + (1 - \alpha)C$ for a trip from $p_1$ to $p_k$ with fuel stops permitted at any of the consecutive service stations between $p_1$ and $p_j$. You
should assume that $j < k$. Moreover, we must always purchase a fuel pack at $p_1$ to start the trip and this counts as one fuel stop. (Notice that once you’ve written this function, we can call it with $\text{optimize}([p_1, \ldots, p_{n-1}], p_n)$ to get our desired solution!)

Your pseudo-code will need to use the 100 mile travel limit per fuel pack and will need to refer to the $d_i$ values which give the distances from $p_1$ to $p_i$ and the $c_i$ values which specify the cost of a fuel pack at point $p_i$. Finally, if there is no possible way to get from $p_1$ to $p_k$ due to the distances between the given fuel stations, the function should return the value $\infty$.

(c) Describe how your algorithm could be converted into a dynamic programming algorithm. In particular:

i. Describe what the dynamic programming table looks like and its dimensions.

ii. Describe the order in which the cells in the table are filled in and the rule used to fill in each cell. Be sure to explain how to fill in the “easy” cells at the beginning as well as the rule for filling in the other cells.

(d) What is the running time of your dynamic programming algorithm? Explain very briefly.