CS155: Modeling

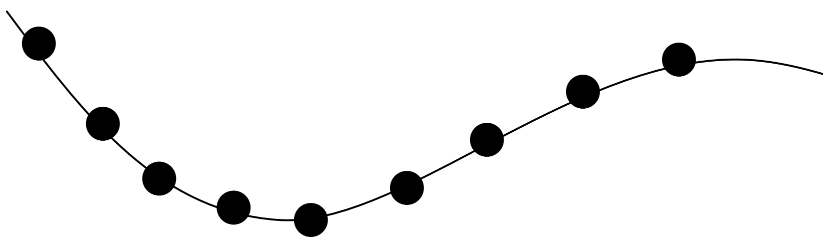
Curves

Z Sweedyk

Overview

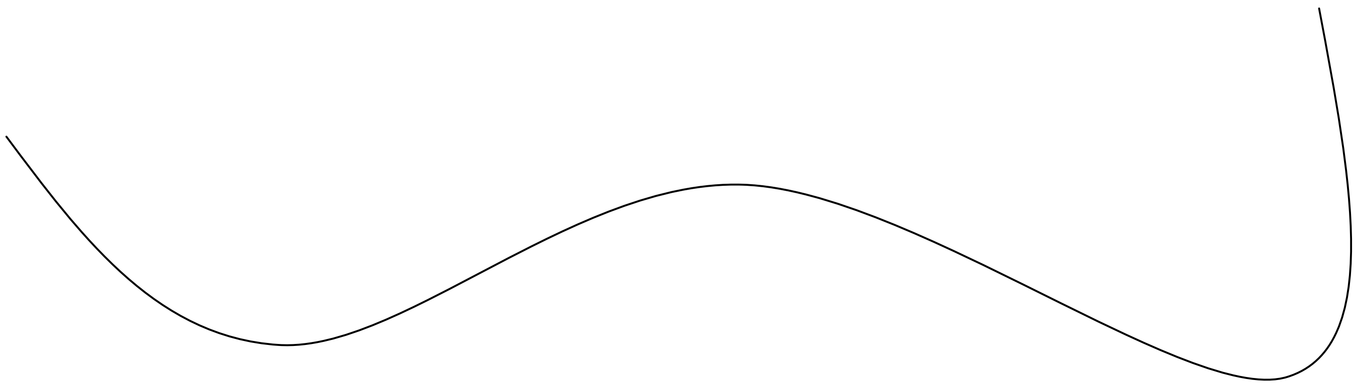
- Curves
  - interpolating curves
  - hermitian splines
  - bezier
  - b-splines
- Surfaces
  - splines
  - nurbs
  - surface subdivision

Drawing Curves



Approximate by line segments between sample points

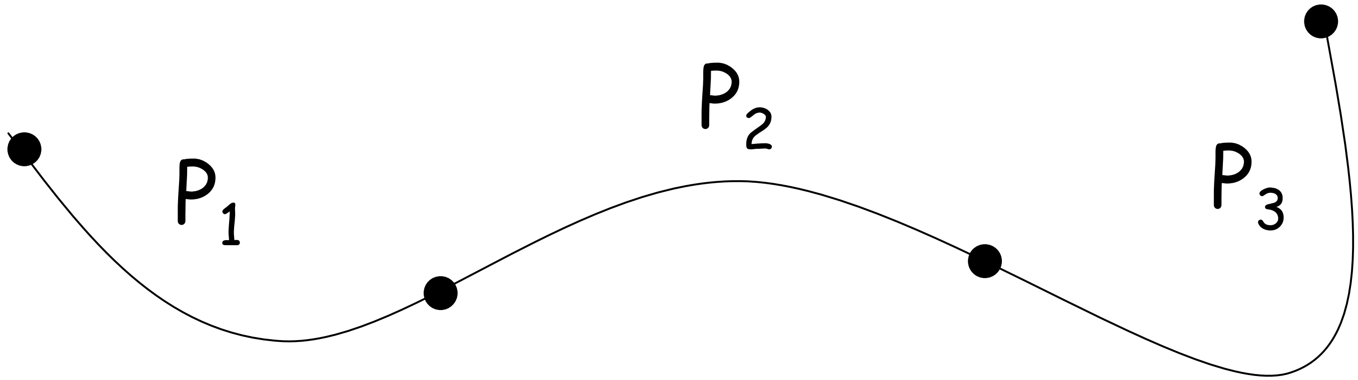
Representing Curves



How should we represent a curve?

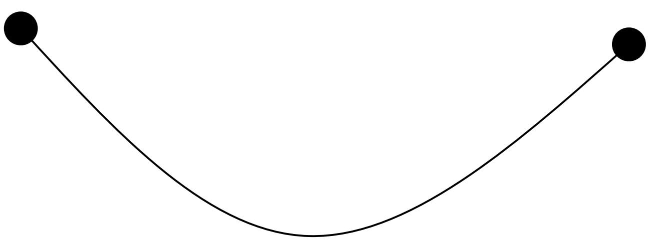
- Flexibility: Can we use the method for a wide range of curves?
- Efficiency: Can we sample it efficiently?
- Usability: Can a user specify it easily?

Complicated Curves



Simple curves connected end-to-end

Simple Curves



How should we represent a simple curve?

- Flexibility
- Efficiency
- Usability
- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

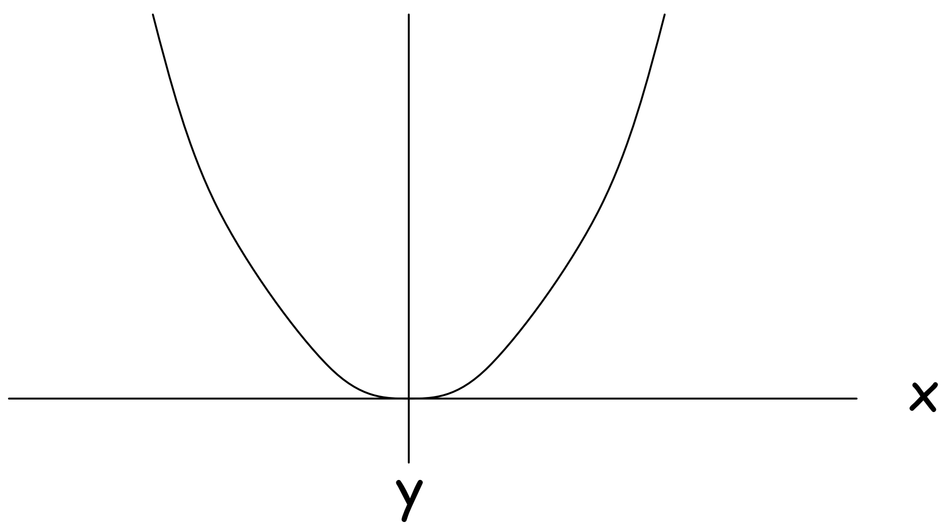
Curve Representation

- Explicit
- Implicit
- Parametric

Explicit

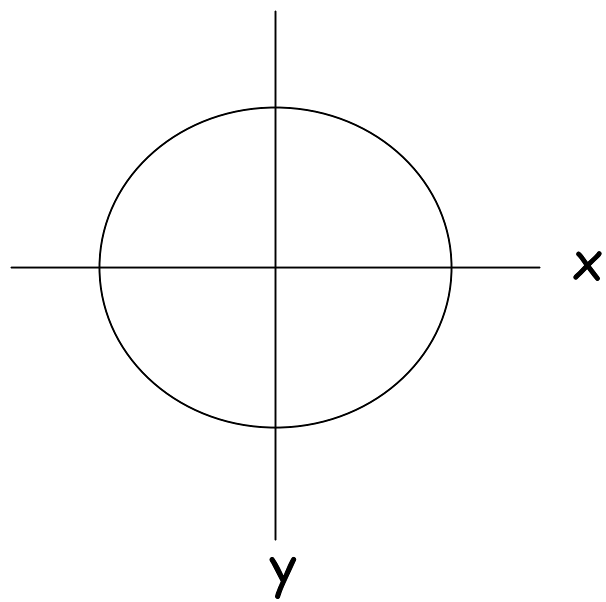
Curve is the trace of a function

Example: $y=x^2/4$



Explicit: flexibility

Many useful curves cannot be represented by explicit functions



Curve Representation

- ~~Explicit~~
- Implicit
- Parametric

Implicit

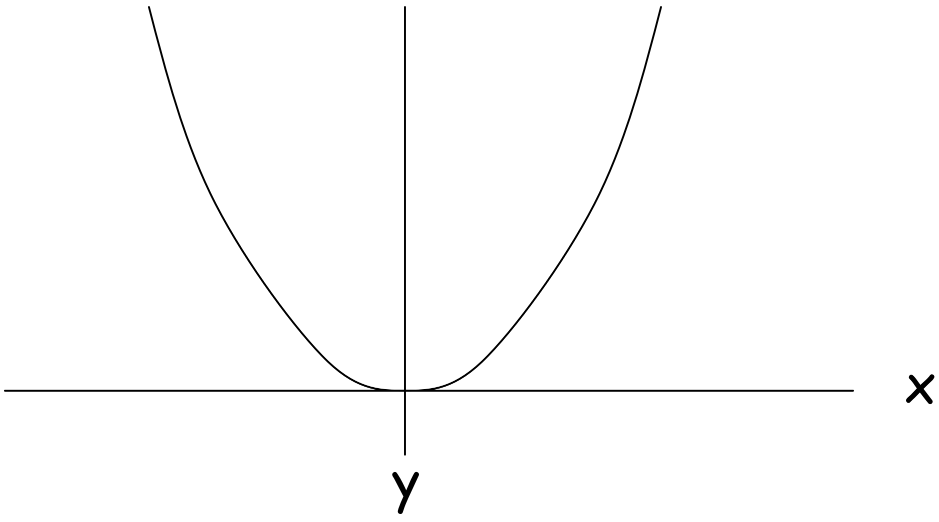
Curve is the zero loci of a function

Example: $f(x,y) = 4y - x^2$

Implicit

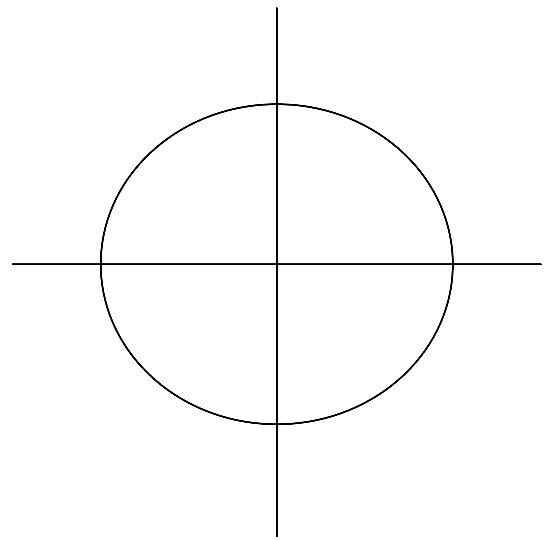
Curve is the zero loci of a function

Example: $f(x,y) = 4y - x^2$



Implicit: more flexibility

$F(x,y)=x^2+y^2-r^2$



But how could we describe a half circle?

Implicit: Efficiency

How can we find the zero loci of an function $f(x,y)$?

Curve Representation

- ~~Explicit~~
- ~~Implicit~~
- Parametric

Parametric

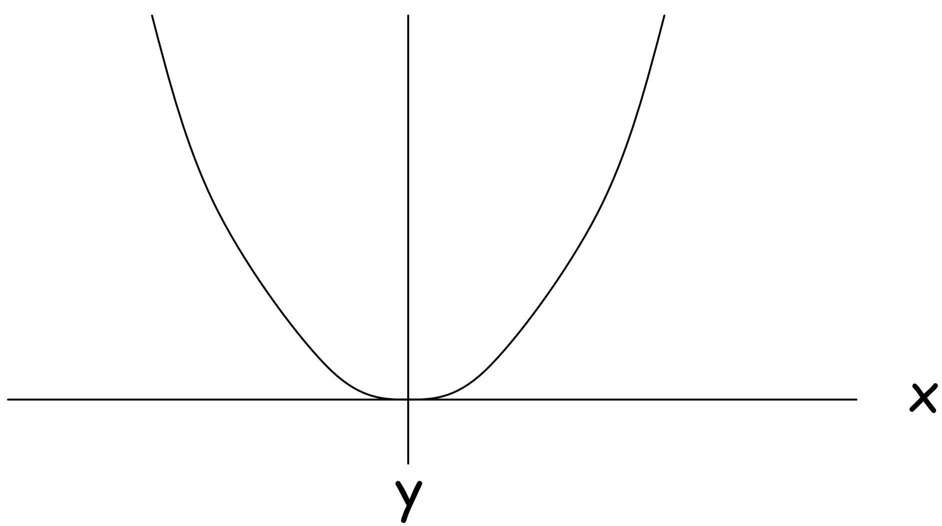
Curve is the range of a function

Example: $x = 2t$, $y = t^2$

Parametric

Curve is the range of a function

Example: $x = 2t,\ y = t^2$



Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
  Minus:
- Usability: not intuitive

Curve Representation

- ~~Explicit~~
- ~~Implicit~~
- ~~Parametric~~

OK -- I give up!

Curve Representation

- ~~Explicit~~
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Parametric: tradeoffs

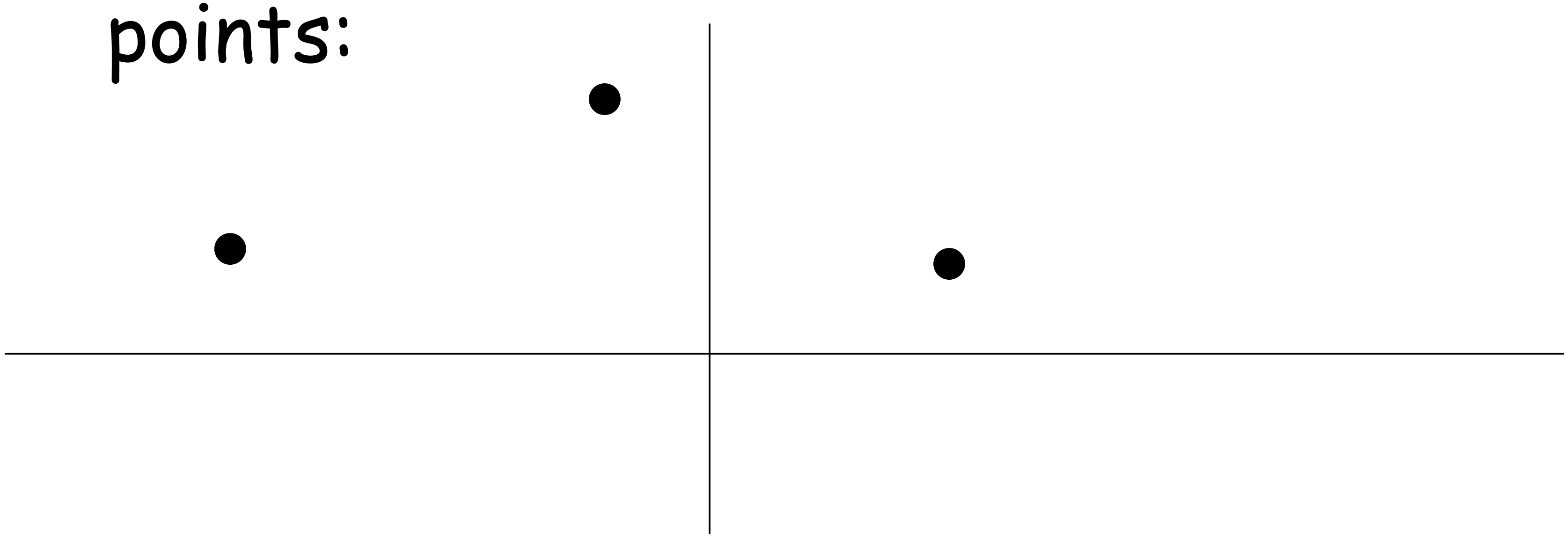
- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
Minus:
- Usability: not intuitive without modeling tools!

Parametric cubic polynomials

- Polynomials are expressive and can be efficiently computed
- Lower degree polynomials can't express non-planar curves
- Higher degree polynomials
  - Wiggle
  - Computationally more expensive

Interpolating polynomials

- Give me the lowest possible degree polynomial curve through these points:



Interpolation

- Points: $(x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)$
- Compute: Quadratic polynomials $x(t)$, $y(t)$, $z(t)$ such that

$$(x(i),y(i),z(i)) = (x_i,y_i,z_i)$$

for $i=0,1,2$

Step 1

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - otherwise it can be anything

Step 1

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$$x(t)=t(t-1)$$

Step 2

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - $x(2) = 1$

Step 2

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - $x(2) = 1$

$$x(t)=t(t-1)/2$$

Step 3

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - $x(2) = x_2$

Step 3

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - $x(2) = x_2$

$$x(t)=x_2 t(t-1)/2$$

Step 4

- Give me a quadratic polynomial x(t) such that:
  - x(t) = 0 if t=0 or t=2
  - $x(1) = x_1$

Step 4

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=2$
  - $x(1) = x_1$

$$x(t)=-x_1 t(t-2)$$

Step 5

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=1$ or $t=2$
  - $x(0) = x_0$

Step 5

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=1$ or $t=2$
  - $x(0) = x_0$

$$x(t)=-x_0(t-1)(t-2)/3\ )$$

Step 6

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(0) = x_0$
  - $x(1) = x_1$
  - $x(2) = x_2$

Step 6

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(0) = x_0$
  - $x(1) = x_1$
  - $x(2) = x_2$

$$x(t)=-x_0(t-1)(t-2)/3 -x_1t(t-2) + x_2t(t-1)/2$$

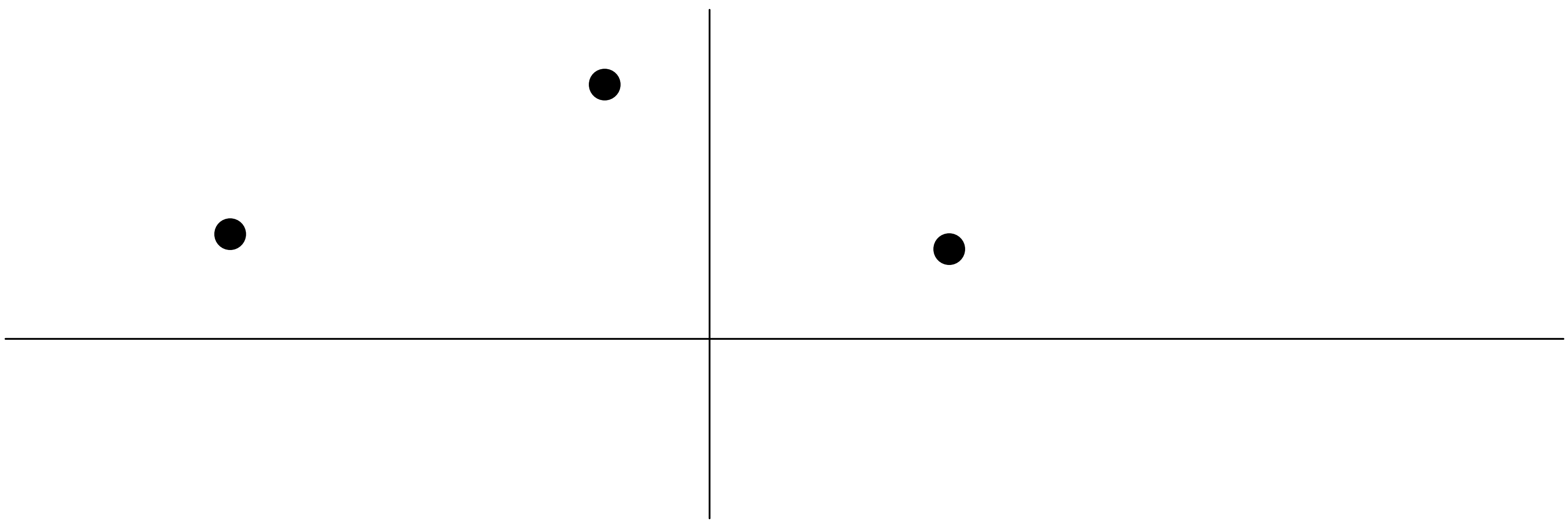
Exercise

Give me a quadratic polynomial $y(t)$ such that

$y(0)=3, y(1/2)=4, y(1)=0$

What about boundary conditions?

- Give me a polynomial curve through these points:

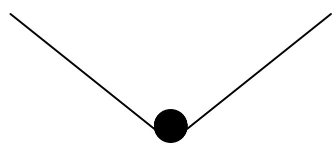


Parametric Continuity

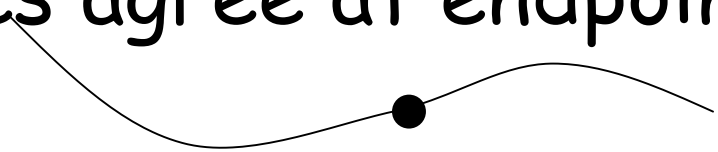
C^i: The 0^{th}, 1^{st}, 2^{nd}, ..., i^{th} derivative of adjacent curves agree at their connecting endpoints.

Parametric Continuity

- C^0: adjacent curves connect at endpoints



- C^1: the 1st derivative of adjacent curves agree at endpoints

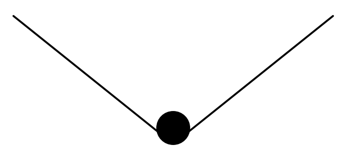


Geometric Continuity

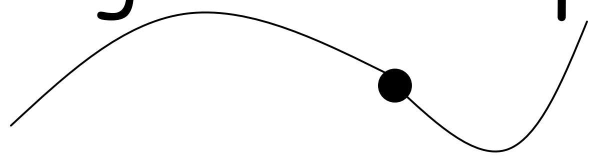
G^i: The 0th, 1st, 2nd, ..., ith derivative of adjacent curves are proportional at endpoints

Geometric Continuity

- G^0: adjacent curves connect at endpoints



- G^1: the 1st derivative of adjacent curves agree at endpoints



Hermitian splines

- Specify endpoint position
- Specify endpoint tangent

Hermitian

- $X(t) = at^3 + bt^2 + ct + d$

- $X(0) = 3$, $X(1) = 2$
- $X'(0) = 1$, $X'(1) = 0$

- Write 4 equations that determine the coefficients a, b, c, and d.

Hermitian: Constraints

- $X(0)$:
- $X(1)$:
- $X'(0)$:
- $X'(1)$:

Hermitian: Constraints

- $X(0)$: $d = 3$
- $X(1)$: $a+b+c+d = 2$
- $X'(0)$: $c = 1$
- $X'(1)$: $3a+2b+c=0$

Hermitian Matrix Form

$$\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$$

Hermitian Matrix Form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Find X(t)

Solve for a, b, c, d: Hint

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hermitian Matrix: X

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 3 \end{bmatrix}$$

Parametric equation: X

$$X(t) = 3t^3 - 5t^2 + t + 3$$

General Matrix: X

$$
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}
$$

Hermitian Matrix: Y

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y(0) \\ Y(1) \\ Y'(0) \\ Y'(1) \end{bmatrix}$$

Hermitian Basis Matrix

2	-2	1	1
-3	3	-2	-1
0	0	1	0
1	0	0	0

Example

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $Y'(0) = 0$, $Y'(1) = 1$

- Write the equations
- Plot the curve for t in [0,1]

Equations

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $X'(0) = 0$, $X'(1) = 1$

- Equations:
- $X(t) = 3t^3 - 5t^2 + t + 3$
- $Y(t) = t^3 - t^2 + 2$

Plot

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $Y'(0) = 0$, $Y'(1) = 1$

Hermitian Description

- Endpoint constraints
- Basis Matrix
- Basis (blending) Functions

Hermitian: X(t)

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

General Matrix: X

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Blending Functions: X

$$X(t) = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Hermitian Blending Functions

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2t^3-3t^2+1, & -2t^3+3t^2, & t^3-2t^2+t, & t^3-t^2 \end{bmatrix}$$

$P_1(t)$ $P_2(t)$ $P_3(t)$ $P_4(t)$

Plot the Hermitian Blending Functions

Hermitian: problem

1. Specifying derivatives is not intuitive for users.
2. Not invariant under affine transformations

Properties of Cubic Bezier Curves

- Control points p_0, p_1, p_2, p_3
- Curve starts at p_0 and ends at p_3.
- Line segments p_o-p_1 and p_3-p_2 are tangent to the curve at, respectively, p_0 and p_3.
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.

Exercise

- Download bezier.cpp from /cs/cs155/labs
- Compile and Play
  - Right click to move red point
  - Left click to select new red point
  - Type “a” to then right click (3 times) to add new control points
  - Type “d” to delete last point