CS155: Modeling

Curves

Z Sweedyk
Overview

• Curves
  - interpolating curves
  - hermitian splines
  - bezier
  - b-splines

• Surfaces
  - splines
  - nurbs
  - surface subdivision
Drawing Curves

Approximate by line segments between sample points
Representing Curves

How should we represent a curve?

• Flexibility: Can we use the method for a wide range of curves?

• Efficiency: Can we sample it efficiently?

• Usability: Can a user specify it easily?
Complicated Curves

Simple curves connected end-to-end
Simple Curves

How should we represent a simple curve?

• Flexibility
• Efficiency
• Usability
• Boundary constraints: Can we specify continuity (including derivatives) at boundaries?
Curve Representation

• Explicit
• Implicit
• Parametric
Explicit

Curve is the trace of a function
Example: $y = \frac{x^2}{4}$
Explicit: flexibility

Many useful curves cannot be represented by explicit functions
Curve Representation

- Explicit
- Implicit
- Parametric
Implicit

Curve is the zero loci of a function
Example: \( f(x,y) = 4y-x^2 \)
Implicit

Curve is the zero loci of a function

Example: \( f(x,y) = 4y-x^2 \)
Implicit: more flexibility

$F(x,y)=x^2+y^2-r^2$

But how could we describe a half circle?
Implicit: Efficiency

How can we find the zero loci of an function $f(x,y)$?
Curve Representation

- Explicit
- Implicit
- Parametric
Parametric

Curve is the range of a function
Example: $x = 2t, \ y = t^2$
Parametric

Curve is the range of a function

Example: $x = 2t$, $y = t^2$
Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify Minus:
- Usability: not intuitive
Curve Representation

- Explicit
- Implicit
- Parametric

OK -- I give up!
Curve Representation

- Explicit
- Implicit
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OK -- I give up!
Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify

Minus:
- Usability: not intuitive without modeling tools!
Parametric cubic polynomials

- Polynomials are expressive and can be efficiently computed

- Lower degree polynomials can’t express non-planar curves

- Higher degree polynomials
  - Wiggle
  - Computationally more expensive
Interpolating polynomials

• Give me the lowest possible degree polynomial curve through these points:
Interpolation

• **Points**: \((x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)\)

• **Compute**: Quadratic polynomials \(x(t), y(t), z(t)\) such that
\[(x(i), y(i), z(i)) = (x_i, y_i, z_i)\]
for \(i = 0, 1, 2\)
Step 1

- Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - otherwise it can be anything
Step 1

• *Give me a quadratic polynomial* $x(t)$ *such that:*
  
  - $x(t) = 0$ if $t=0$ or $t=1$
  - otherwise it can be anything

  $$x(t) = t(t-1)$$
Step 2

• Give me a quadratic polynomial \( x(t) \) such that:
  - \( x(t) = 0 \) if \( t=0 \) or \( t=1 \)
  - \( x(2) = 1 \)
Step 2

• Give me a quadratic polynomial \( x(t) \) such that:
  - \( x(t) = 0 \) if \( t=0 \) or \( t=1 \)
  - \( x(2) = 1 \)

\[
x(t) = \frac{t(t-1)}{2}
\]
Step 3

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=1$
  - $x(2) = x_2$
Step 3

• Give me a quadratic polynomial \( x(t) \) such that:
  - \( x(t) = 0 \) if \( t=0 \) or \( t=1 \)
  - \( x(2) = x_2 \)

\[
x(t) = x_2 \frac{t(t-1)}{2}
\]
Step 4

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=2$
  - $x(1) = x_1$
Step 4

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=0$ or $t=2$
  - $x(1) = x_1$

  $x(t) = -x_1t(t-2)$
Step 5

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=1$ or $t=2$
  - $x(0) = x_0$
Step 5

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(t) = 0$ if $t=1$ or $t=2$
  - $x(0) = x_0$

$$x(t) = -x_0(t-1)(t-2)/3$$
Step 6

• Give me a quadratic polynomial $x(t)$ such that:
  - $x(0) = x_0$
  - $x(1) = x_1$
  - $x(2) = x_2$
Step 6

• Give me a quadratic polynomial \( x(t) \) such that:
  - \( x(0) = x_0 \)
  - \( x(1) = x_1 \)
  - \( x(2) = x_2 \)

\[
x(t) = -\frac{x_0(t-1)(t-2)}{3} - x_1 t(t-2) + \frac{x_2 t(t-1)}{2}
\]
Exercise

Give me a quadratic polynomial $y(t)$ such that

\[ y(0)=3, \ y(1/2)=4, \ y(1)=0 \]
What about boundary conditions?

- Give me a polynomial curve through these points:
Parametric Continuity

$C^i$: The $0^{th}$, $1^{st}$, $2^{nd}$, ..., $i^{th}$ derivative of adjacent curves agree at their connecting endpoints.
Parametric Continuity

- $C^0$: adjacent curves connect at endpoints

- $C^1$: the 1st derivative of adjacent curves agree at endpoints
Geometric Continuity

$G^i$: The $0^{th}$, $1^{st}$, $2^{nd}$, ..., $i^{th}$ derivative of adjacent curves are proportional at endpoints
Geometric Continuity

- $G^0$: adjacent curves connect at endpoints

- $G^1$: the 1st derivative of adjacent curves agree at endpoints
Hermitian splines

• Specify endpoint position
• Specify endpoint tangent
Hermitian

- $X(t) = at^3 + bt^2 + ct + d$

- $X(0) = 3, \quad X(1) = 2$
- $X'(0) = 1, \quad X'(1) = 0$

- Write 4 equations that determine the coefficients $a, b, c,$ and $d$. 
Hermitian: Constraints

- $X(0)$:
- $X(1)$:
- $X'(0)$:
- $X'(1)$:
Hermitian: Constraints

• $X(0): \ d = 3$
• $X(1): \ a+b+c+d = 2$
• $X'(0): \ c = 1$
• $X'(1): \ 3a+2b+c=0$
Hermitian Matrix Form

\[
\begin{pmatrix}
- & - & - & - \\
- & a & b & - \\
- & c & d & - \\
- & - & - & -
\end{pmatrix}
= 
\begin{pmatrix}
- \\
- \\
- \\
-
\end{pmatrix}
\]
Hermitian Matrix Form

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
= \begin{bmatrix}
3 \\
2 \\
1 \\
0
\end{bmatrix}
\]
Find $X(t)$
Solve for a, b, c, d:  Hint

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-1
\end{bmatrix}
= 
\begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Hermitian Matrix: $X$

\[
\begin{pmatrix}
a \\ b \\ c \\ d
\end{pmatrix}
= 
\begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
3 \\ 2 \\ 1 \\ 0
\end{pmatrix}
= 
\begin{pmatrix}
3 \\ -5 \\ 1 \\ 3
\end{pmatrix}
\]
Parametric equation: $X$

$$X(t) = 3t^3 - 5t^2 + t + 3$$
General Matrix: $X$

$$
\begin{bmatrix}
  a & b & c & d \\
  X(0) & X(1) & X'(0) & X'(1)
\end{bmatrix}
= \begin{bmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
$$
Hermitian Matrix: $Y$

\[
\begin{pmatrix}
a & b \\
b & c \\
c & d \\
d & \\
\end{pmatrix} = 
\begin{pmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix} 
\begin{pmatrix}
Y(0) \\
Y(1) \\
Y'(0) \\
Y'(1) \\
\end{pmatrix}
\]
Hermitian Basis Matrix

\[
\begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Example

• $X(0) = 3, \ X(1) = 2, \ X'(0) = 1, \ X'(1) = 0$
• $Y(0) = 2, \ Y(1) = 2, \ Y'(0) = 0, \ Y'(1) = 1$

• Write the equations
• Plot the curve for $t$ in $[0,1]$
Equations

• $X(0) = 3, \quad X(1) = 2, \quad X'(0) = 1, \quad X'(1) = 0$
• $Y(0) = 2, \quad Y(1) = 2, \quad X'(0) = 0, \quad X'(1) = 1$

• Equations:
  • $X(t) = 3t^3 - 5t^2 + t + 3$
  • $Y(t) = t^3 - t^2 + 2$
Plot

- \( X(0) = 3, \ X(1) = 2, \ X'(0) = 1, \ X'(1) = 0 \)
- \( Y(0) = 2, \ Y(1) = 2, \ Y'(0) = 0, \ Y'(1) = 1 \)
Hermitian Description

• Endpoint constraints
• Basis Matrix
• Basis (blending) Functions
Hermitian: $X(t)$

$X(t) = \begin{pmatrix}
  t^3 & t^2 & t & 1 \\
  a & b & c & d
\end{pmatrix}$
General Matrix: \( X \)

\[
X(t) = \begin{bmatrix}
    t^3 & t^2 & t & 1 \\
    2 & -2 & 1 & 1 \\
    -3 & 3 & -2 & -1 \\
    0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix}
\]

\( X(0) \)

\( X(1) \)

\( X'(0) \)

\( X'(1) \)
Blending Functions: \( X \)

\[
X(t) = \begin{pmatrix}
P_1(t) & P_2(t) & P_3(t) & P_4(t)
\end{pmatrix}
\begin{pmatrix}
X(0) \\
X(1) \\
X'(0) \\
X'(1)
\end{pmatrix}
\]
Hermitian Blending Functions

\[
\begin{bmatrix}
  t^3 & t^2 & t & 1 \\
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
\end{bmatrix} =
\begin{bmatrix}
  2t^3-3t^2+1, & -2t^3+3t^2, & t^3-2t^2+t, & t^3-t^2 \\
  P_1(t) & P_2(t) & P_3(t) & P_4(t) \\
\end{bmatrix}
\]
Plot the Hermitian Blending Functions
Hermitian: problem

1. Specifying derivatives is not intuitive for users.
2. Not invariant under affine transformations
Properties of Cubic Bezier Curves

- Control points \( p_0, p_1, p_2, p_3 \)
- Curve starts at \( p_0 \) and ends at \( p_3 \).
- Line segments \( p_0-p_1 \) and \( p_3-p_2 \) are tangent to the curve at, respectively, \( p_0 \) and \( p_3 \).
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.
Exercise

• Download bezier.cpp from /cs/cs155/labs

• Compile and Play
  - Right click to move red point
  - Left click to select new red point
  - Type “a” to then right click (3 times) to add new control points
  - Type “d” to delete last point