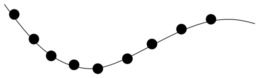
CS155: Modeling

More Curves
Z Sweedyk

Overview

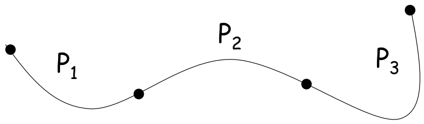
- Curves
  - interpolating curves
  - hermitian splines
  - catmull-rom
  - **bezier**
  - b-splines
- Surfaces
  - splines
  - nurbs
  - surface subdivision

Drawing Curves



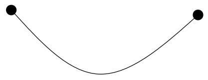
Approximate by line segments between sample points

Complicated Curves



Simple curves connected end-to-end

Simple Curves



How should we represent a simple curve?

- Flexibility
- Efficiency
- Usability
- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

Curve Representation

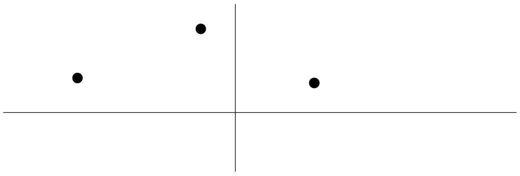
- Explicit
- Implicit
- **Parametric**

Interpolation

- Points: (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)
- Compute: Quadratic polynomials $x(t)$, $y(t)$, $z(t)$ such that
  $$(x(i),y(i),z(i)) = (x_i,y_i,z_i)$$
  for $i=0,1,2$
- Solution for $x(t)=\Sigma_i\, x_i \left[\Pi_{j\neq i}\, (x-x_j)/(x_i-x_j)\right]$

What about boundary conditions?

- *Give me a polynomial curve through these points:*



Parametric Continuity

C^i: The 0th, 1st, 2nd, …, ith derivative of adjacent curves agree at their connecting endpoints.

Geometric Continuity

G^i: The 0th, 1st, 2nd, …, ith derivative of adjacent curves are proportional at endpoints

Hermitian splines

- Specify endpoint position
- Specify endpoint tangent

Hermitian Basis Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix} \leftarrow \text{input}$$

$\hookrightarrow X(t)=at^3 + bt^2 + ct + d$

Hermitian Blending Functions: X

$$X(t) = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Hermitian Blending Functions

$$\begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix} =$$

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2t^3-3t^2+1, & -2t^3+3t^2, & t^3-2t^2+t, & t^3-t^2 \end{bmatrix}$$

Hermitian: problem

1. Specifying derivatives is not intuitive for users.
2. Not invariant under affine transformations

Derivative specification

- Hermitian: Enforce continuity constraints
- Catmull-Rom splines

Exercise

- First Hermitian curve $x(t)$:
  - $x_1(0)$, $x_1(1)$, $x'_1(0)$, $x'_1(1)$
- Second Hermitian curve:
  - What conditions provide C^i & G^i continuity for $i=0,1$?

Hermitian: C^0 & G^0

$x_2(0)=x_1(1)$

Hermitian: C^1 & G^1

C^1: $x_2(0)=x_1(1)$ and $x_2'(0)=x_1'(1)$

G^1: $x_2(0)=x_1(1)$ and $x_2'(0)=\alpha x_1'(1)$ for some α

(note: same α factor applies to $y(t)$ and $z(t)$)

Enforcing continuity

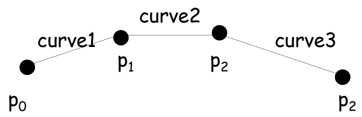
- First Hermitian curve:
  - $x_1(0)$, $x_1(1)$, $x'_1(0)$, $x'_1(1)$
- Second Hermitian curve:
  - What conditions provide C^2 continuity?

Hermitian: C^2 Constraints

- $X_2(t) = a_2t^3+b_2t^2+c_2t+d_2$

- $X_2(0)=X_1(1)$: $d_2=X_1(1)$
- $X_2'(0)=X_1'(1)$: $3a_2+2b_2+c_2=X_1'(1)$
- $X_2''(0)=X_1''(1)$: $6a_2+2b_2=X_1''(1)$
- $X_2(1)$: user specifies

Catmull-Rom Spline: C^1

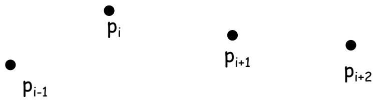


tangent at p_1 = (1/2) $\langle p_0,p_1\rangle$

tangent at p_2 = (1/2) $\langle p_1,p_2\rangle$

Catmull-Rom Basis Matrix

- Compute the basis matrix for the Catmull-Rom spline from p_i to p_{i+1}



Catmull-Rom constraints

$X(t)) = at^3 + bt^2 + ct + d$

assume $p_i=(x_i,y_i)$

- $X(0)=d=x_i$
- $X(1)=a+b+c=x_{i+1}$
- $X'(0)=c=(x_{i+1}-x_i)/2$
- $X'(1)=6a+2b=(x_{i+2}-x_{i+1})$

Catmull-Rom Basis Matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -.5 & 0 & .5 & 0 \\ 0 & -.5 & 0 & .5 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \end{bmatrix}$$

Catmull-Rom Basis Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \end{bmatrix}$$

Hermitian/Catmull-Rom: problem

Not invariant under affine transformations

In other words: transforming (rotate, scale, translate) the control points does not yield the transformed curve

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Properties of Cubic Bezier Curves

- Control points p_0, p_1, p_2, p_3
- Curve starts at p_0 and ends at p_3.
- Line segments p_o-p_1 and p_3-p_2 are tangent to the curve at, respectively, p_0 and p_3.
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.

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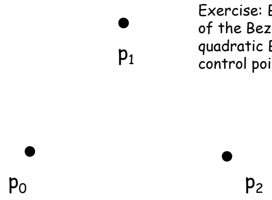
deCasteljau's Algorithm for quadratic bezier in 2D



Exercise: Based on the properties of the Bezier curve, sketch the quadratic Bezier in 2d for the control points shown

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Exercise

- Download bezier.exe from the course web page.
- Run
  - Right click to move red point
  - Left click to select new red point
  - Type "a" to then right click (3 times) to add new control points
  - Type "d" to delete last point

Bezier Basis Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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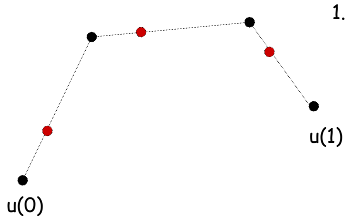
deCasteljau's algorithm: compute u(1/3)



1. compute 1/3 point on control lines

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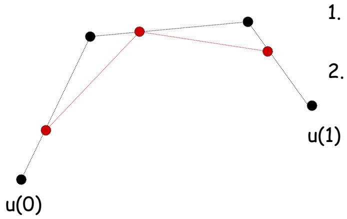
deCasteljau's algorithm: compute u(1/3)



1. compute 1/3 point on control lines
2. connect

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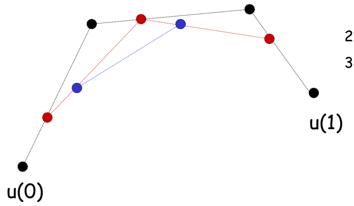
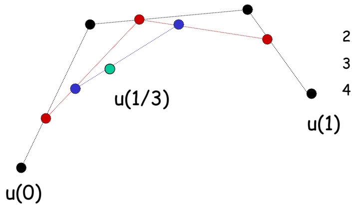
deCasteljau's algorithm: compute u(1/3)



1. compute 1/3 point on control lines
2. connect
3. repeat

---

deCasteljau's algorithm: compute u(1/3)



1. compute 1/3 point on control lines
2. connect
3. repeat
4. repeat

---

exercise

- Use deCasteljau's algorithm to compute U(1/2) for the previous example.

why does this work?

Linear Space

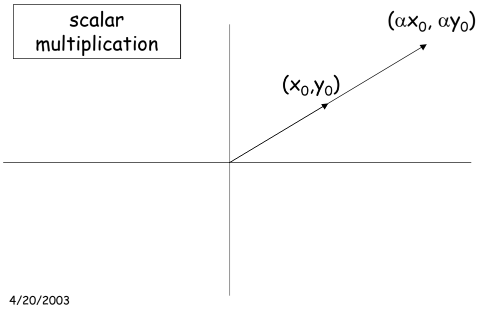
- Vectors and Scalars
- Vector addition
- Scalar multiplication
- Properties:
  - Addition is commutative and associative
  - Multiplication is associative
  - Multiplication distributes of addition
  - Additive and multiplicative identities
  - Additive identity and inverse
  - Multiplicative identity

Linear Space

Our favorite linear space if $\Re^n$.

$\Re^2$

scalar multiplication



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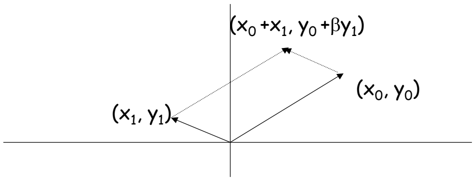
vector addition

$\Re^2$



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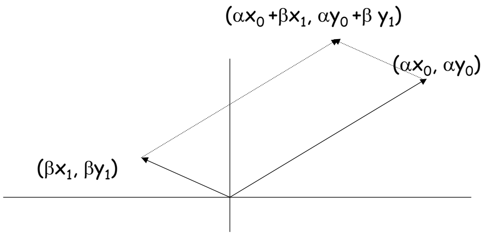
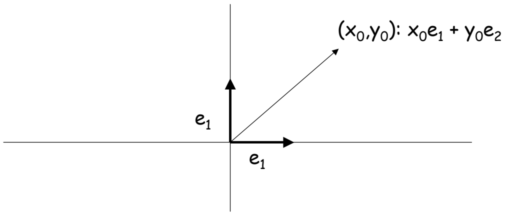
linear combination

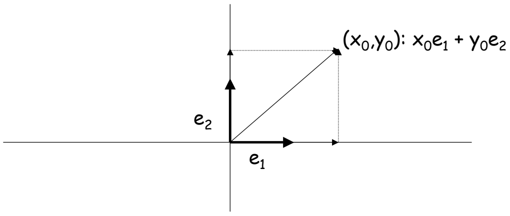
$\Re^2$



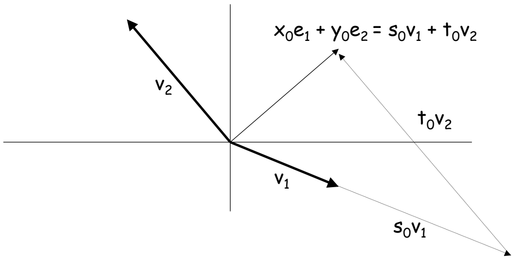
Standard Basis for $\Re^2$



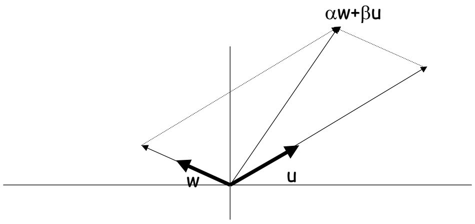
Standard Basis for $\Re^2$



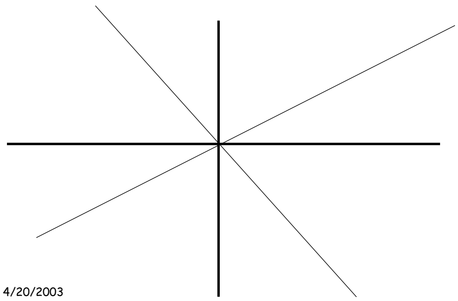
Basis for $\Re^2$



Linear combinations are basis independent!!



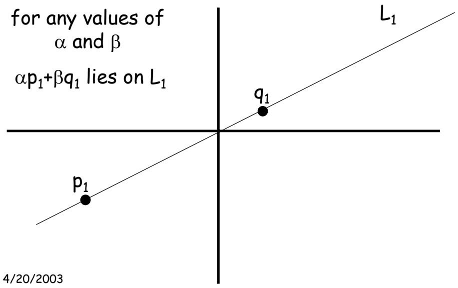
Linear Subspaces



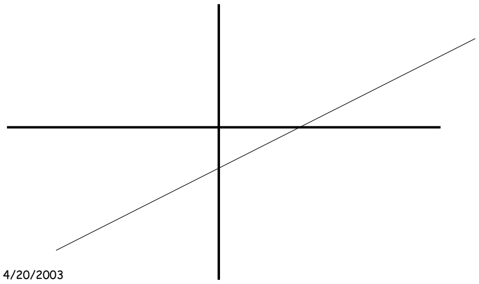
Linear combination

for any values of α and β

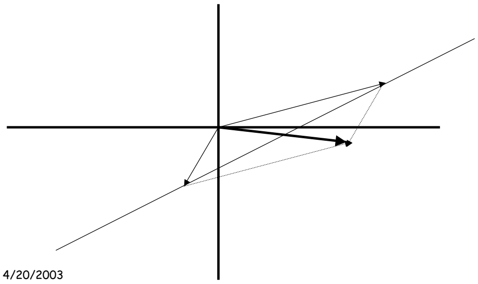
$\alpha p_1+\beta q_1$ lies on L_1



Linear Subspace, NOT



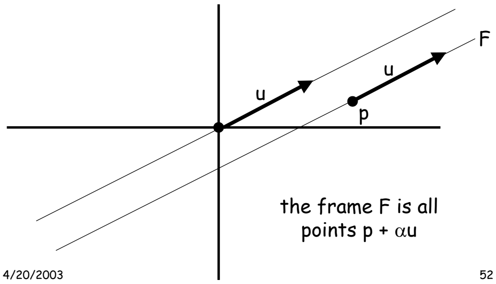
Linear Subspace, NOT



Frame

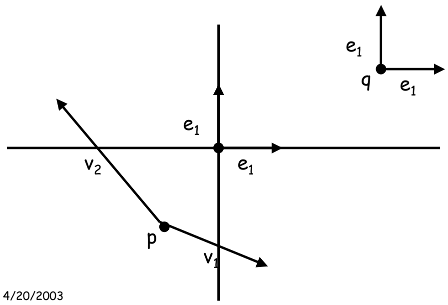
Point and basis vectors

1D Frames



the frame F is all points p + αu

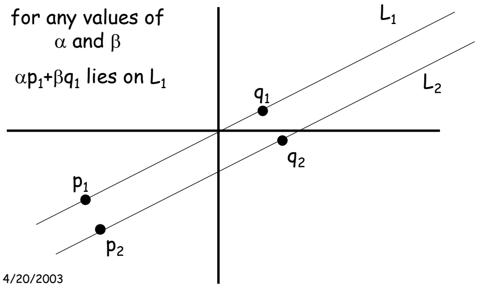
2D Frames



Linear combination of points

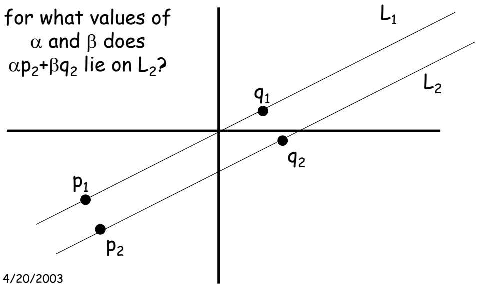
for any values of
α and β

$\alpha p_1+\beta q_1$ lies on L_1



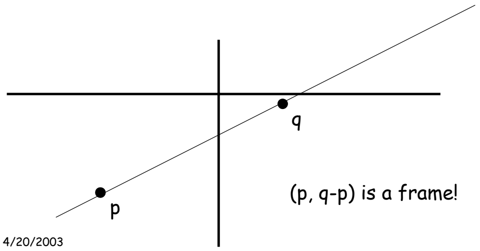
Linear combination of points

for what values of α and β does $\alpha p_2+\beta q_2$ lie on L_2?



Linear combination of points

if $\alpha+\beta=1$ then $\alpha p+\beta q = p + (\alpha-1)p + \beta q = p + \beta(q-p)$



(p, q-p) is a frame!

Affine combination

- linear combination of points with weights that add to 1
- affine combinations are frame independent!

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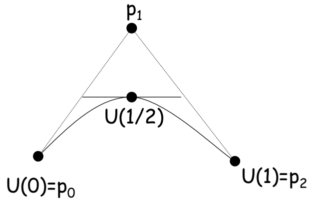
quadratic bezier U(t)=(X(t),Y(t))



Polar form

Given: $X(t)=at^2 + bt + c$

Construct: $x(t_1,t_2)$ such that

- $x(t,t)=X(t)$
- $x(t_1,t_2)= x(t_2,t_1)$

Polar form

Given: $X(t)=at^2 + bt + c$

Construct: $x(t_1,t_2)$ such that

- $x(t,t)=X(t)$
- $x(t_1,t_2)= x(t_2,t_1)$

$x(t_1,t_2)= a\, t_1t_2 + (b/2)(t_1 + t_2) + c$

Exercise

Given: $X(t)=at^3 + bt^2 + ct + d$

Construct the polar form $x(t_1,t_2,t_3)$

Linear Functions

f(u) is linear if $f(\alpha u_1 + \beta u_2)=\alpha f(u_1)+\beta f(u_2)$

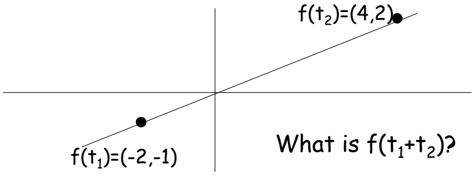
Note: for any linear function
$f(0)=f(u-u)=f(u)-f(u)=0$

Linear functions

$f(t) = \alpha t$ is linear

Linear Functions

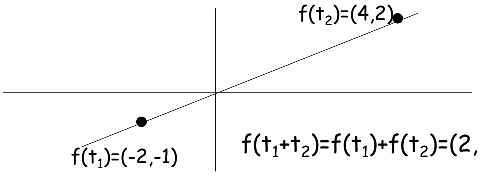
Suppose $f(t)$: $\Re \rightarrow \Re^2$ is a linear function and $f(t_1)=(-2,-1)$ and $f(t_2)=(3,2)$.



What is $f(t_1+t_2)$?

Linear Functions

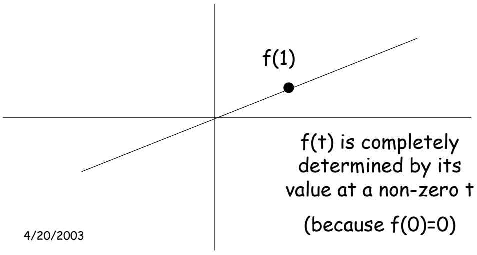
Suppose $f(t)$: $\Re \rightarrow \Re^2$ is a linear function and $f(t_1)=(-2,-1)$ and $f(t_2)=(3,2)$.



$f(t_1+t_2)=f(t_1)+f(t_2)=(2,1)$

Linear Functions

Suppose f(1)=(2,1).
What is f(t)?

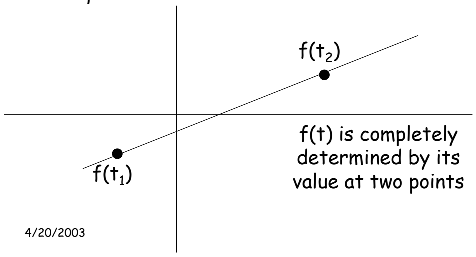


f(t) is completely
determined by its
value at a non-zero t

(because f(0)=0)

Affine Functions

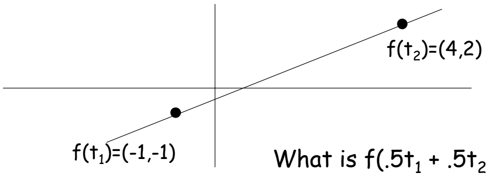
f(t) is affine if $f(\alpha t_1 + \beta t_2) = \alpha f(t_1) + \beta f(t_2)$ when $\alpha + \beta = 1$



f(t) is completely determined by its value at two points

Affine Functions

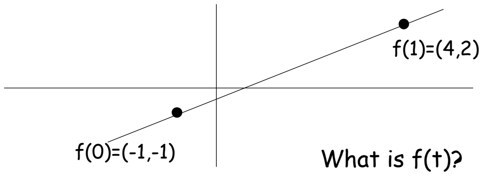
Suppose f(t): $\mathfrak{R} \rightarrow \mathfrak{R}^2$ is an affine function such that $f(t_1)=(-1,-1)$ and $f(t_2)=(4,2)$.



What is $f(.5t_1 + .5t_2)$?

Affine Functions

Suppose f(t): $\mathfrak{R} \rightarrow \mathfrak{R}^2$ is an affine function such that f(0)=(-1,-1) and f(1)=(4,2).



What is f(t)?

Linear & Affine

- f(t) = at is linear
- f(t) = at+b is affine

Multi-Affine Functions

$f(u_1,u_2,\ldots,u_n)$ is multi-affine if holding all but one parameter fixed yields an affine function

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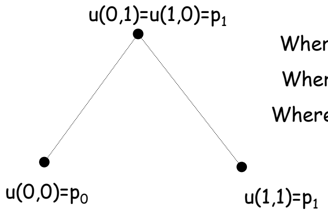
$u(t_1,t_2)$ is a multi-affine, symmetric function

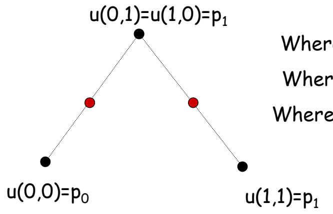


Where is u(0.5,0)?

Where is u(0.5,1)?

Where is u(0.5,0.5)?

$u(t_1,t_2)$ is a multi-affine, symmetric function



Where is u(0.5,0)?

Where is u(0.5,1)?

Where is u(0.5,0.5)?

Polar form

Given: $X(t)=at^2 + bt + c$

Construct: $x(t_1,t_2)$ such that

- $x(t,t)=X(t)$
- $x(t_1,t_2)= x(t_2,t_1)$

$$x(t_1,t_2)= a\, t_1t_2 + (b/2)(t_1 + t_2) + c$$

$\boxed{x(t_1,t_2) \text{ is a multi-affine function}}$

Polar form

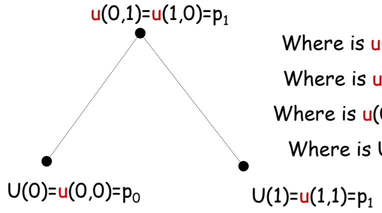
$$x(t_1,t_2)= a\, t_1t_2 + (b/2)(t_1 + t_2) + c$$

Hold t_1 fixed:

$$x'(t_2)= \underbrace{(at_1 + b/2)t_2}_{\text{scalar multiplication}} + \underbrace{(b/2)t_1 + c}_{\text{constant}}$$

similarly for t_2

$u(t_1,t_2)$ is the polar form of $U(t)$



Where is u(0.5,0)?

Where is u(0.5,1)?

Where is u(0.5,0.5)?

Where is U(0.5)?

Cubic Bezier

Where is U(1/2)?