Overview

- Curves
  - interpolating curves
  - hermitian splines
  - catmull-rom
  - bezier
  - b-splines
- Surfaces
  - splines
  - nurbs
  - surface subdivision

Drawing Curves

Approximate by line segments between sample points

Complicated Curves

Simple curves connected end-to-end

Simple Curves

How should we represent a simple curve?

- Flexibility
- Efficiency
- Usability
- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

Curve Representation

- Explicit
- Implicit
- Parametric
Interpolation

• Points: \((x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)\)

• Compute: Quadratic polynomials \(x(t), y(t), z(t)\) such that
  \((x(i), y(i), z(i)) = (x_i, y_i, z_i)\)
  for \(i=0,1,2\)

• Solution for \(x(t) = \sum_i x_i [\prod_{j \neq i} (x-x_j)/(x_i-x_j)]\)

What about boundary conditions?

• Give me a polynomial curve through these points:

Geometric Continuity

\(G^i\): The 0\(^{th}\), 1\(^{st}\), 2\(^{nd}\), …, \(i^{th}\) derivative of adjacent curves are proportional at endpoints

Hermitian splines

• Specify endpoint position
• Specify endpoint tangent

Hermitian Basis Matrix

\[
\begin{array}{c|cccc}
0 & 2 & -2 & 1 & 1 \\
1 & -3 & 3 & -2 & -1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\(X(t) = at^3 + bt^2 + ct + d\)
Hermitian Blending Functions: $X$

$$X(t) = \begin{bmatrix} P_0(t) & P_1(t) & P_2(t) & P_3(t) \\ \end{bmatrix}$$

Hermitian: problem

1. Specifying derivatives is not intuitive for users.
2. Not invariant under affine transformations

Exercise

- First Hermitian curve $x(t)$:
  - $x_1(0)$, $x_1(1)$, $x'_1(0)$, $x'_1(1)$
- Second Hermitian curve:
  - What conditions provide $C^i$ continuity for $i=0,1$?

Derivative specification

- Hermitian: Enforce continuity constraints
- Catmull-Rom splines

Hermitian: $C^0 & G^0$

$$x_2(0)=x_1(1)$$
Hermitian: $C^1$ & $G^1$

$C^1$: $x_2(0) = x_1(1)$ and $x_2'(0) = x_1'(1)$

$G^1$: $x_2(0) = x_1(1)$ and $x_2'(0) = \alpha x_1'(1)$ for some $\alpha$

(note: same $\alpha$ factor applies to $y(t)$ and $z(t)$)

Enforcing continuity

- First Hermitian curve:
  - $x_1(0)$, $x_1(1)$, $x_1'(0)$, $x_1'(1)$
- Second Hermitian curve:
  - What conditions provide $C^2$ continuity?

Hermitian: $C^2$ Constraints

- $X_2(t) = a_2 t^3 + b_2 t^2 + c_2 t + d_2$
- $X_2(0) = X_1(1)$: $d_2 = X_1(1)$
- $X_2'(0) = X_1'(1)$: $3a_2 + 2b_2 + c_2 = X_1'(1)$
- $X_2''(0) = X_1''(1)$: $6a_2 + 2b_2 = X_1''(1)$
- $X_2(1)$: user specifies

Catmull-Rom Spline: $C^1$

- Tangent at $p_1 = (1/2) <p_0, p_1>$
- Tangent at $p_2 = (1/2) <p_1, p_2>$

Catmull-Rom Basis Matrix

- Compute the basis matrix for the Catmull-Rom spline from $p_i$ to $p_{i+1}$

Catmull-Rom constraints

$X(t) = at^3 + bt^2 + ct + d$

- $X(0) = d = x_i$
- $X(1) = a + b + c = x_{i+1}$
- $X'(0) = c = (x_{i+1} - x_i) / 2$
- $X'(1) = 6a + 2b = (x_{i+2} - x_{i+1})$
**Catmull-Rom Basis Matrix**

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
6 & 2 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
-5 & 0 & 5 & 0 \\
0 & -5 & 0 & 5 \\
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_i+1 \\
x_i+2 \\
x_i+3 \\
\end{bmatrix}
\]

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**Hermitian/Catmull-Rom: problem**

Not invariant under affine transformations

In other words: transforming (rotate, scale, translate) the control points does not yield the transformed curve

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**Properties of Cubic Bezier Curves**

- Control points \( p_0, p_1, p_2, p_3 \)
- Curve starts at \( p_0 \) and ends at \( p_3 \).
- Line segments \( p_0-p_1 \) and \( p_3-p_2 \) are tangent to the curve at, respectively, \( p_0 \) and \( p_3 \).
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.

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**deCasteljau’s Algorithm for quadratic bezier in 2D**

Exercise: Based on the properties of the Bezier curve, sketch the quadratic Bezier in 2d for the control points shown.

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**Exercise**

- Download bezier.exe from the course web page.
- Run
  - Right click to move red point
  - Left click to select new red point
  - Type “a” to then right click (3 times) to add new control points
  - Type “d” to delete last point

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Bezier Basis Matrix

\[
\begin{bmatrix}
 a \\
 b \\
 c \\
 d
\end{bmatrix} = \begin{bmatrix}
 1 & 3 & -3 & 1 \\
 3 & -6 & 3 & 0 \\
 -3 & 3 & 0 & 0 \\
 1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
 x_0 \\
 x_1 \\
 x_2 \\
 x_3
\end{bmatrix}
\]

deCasteljau's algorithm: compute \( u(1/3) \)

1. compute 1/3 point on control lines
2. connect
3. repeat
4. repeat

exercise

- Use deCasteljau's algorithm to compute \( U(1/2) \) for the previous example.
why does this work?

Linear Space

- Vectors and Scalars
- Vector addition
- Scalar multiplication
- Properties:
  - Addition is commutative and associative
  - Multiplication is associative
  - Multiplication distributes of addition
  - Additive and multiplicative identities
  - Additive identity and inverse
  - Multiplicative identity

Our favorite linear space is $\mathbb{R}^n$.

For $\mathbb{R}^2$:
- Scalar multiplication:
  $$(x_0, y_0) \rightarrow (\alpha x_0, \alpha y_0)$$
- Vector addition:
  $$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + \beta y_1)$$
- Linear combination:
  $$(\alpha x_0 + \beta x_1, \alpha y_0 + \beta y_1)$$
Standard Basis for $\mathbb{R}^2$

- $e_1$ and $e_2$

$(x_0, y_0): x_0e_1 + y_0e_2$

Basis for $\mathbb{R}^2$

- $v_1$ and $v_2$

$x_0v_1 + y_0v_2 = s_0v_1 + t_0v_2$

Linear combinations are basis independent!!

Linear Subspaces

- Linear combination $p_1 + \alpha q_1$ lies on $L_1$

for any values of $\alpha$ and $\beta$
Frame

Point and basis vectors

2D Frames

Linear combination of points

for any values of $\alpha$ and $\beta$ $\alpha p_1 + \beta q_1$ lies on $L_1$
Linear combination of points

for what values of $\alpha$ and $\beta$ does $\alpha p_2 + \beta q_2$ lie on $L_2$?

$L_1$  $L_2$

$p_1$ $q_1$

$p_2$ $q_2$

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Affine combination

• linear combination of points with weights that add to 1
• affine combinations are frame independent!

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Polar form

Given: $X(t) = at^2 + bt + c$
Construct: $x(t_1, t_2)$ such that
• $x(t, t) = X(t)$
• $x(t_1, t_2) = x(t_2, t_1)$

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linear combination of points

if $\alpha + \beta = 1$ then $\alpha p + \beta q = p + (\alpha - 1)p + \beta(q - p)$

$(p, q - p)$ is a frame!

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quadratic bezier $U(t) = (X(t), Y(t))$

$U(0) = p_0$

$p_1$

$U(1/2)$

$U(1) = p_2$

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Polar form

Given: $X(t) = at^2 + bt + c$
Construct: $x(t_1, t_2)$ such that
• $x(t, t) = X(t)$
• $x(t_1, t_2) = x(t_2, t_1)$

$x(t_1, t_2) = a t_1 t_2 + (b/2)(t_1 + t_2) + c$

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Exercise

Given: \( X(t) = at^3 + bt^2 + ct + d \)

Construct the polar form \( x(t_1, t_2, t_3) \)

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Linear Functions

\( f(u) \) is linear if \( f(\alpha u_1 + \beta u_2) = \alpha f(u_1) + \beta f(u_2) \)

Note: for any linear function \( f(0) = f(u-u) = f(u) - f(u) = 0 \)

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Linear functions

\( f(t) = \alpha t \) is linear

---

Suppose \( f(t): \mathbb{R} \rightarrow \mathbb{R}^2 \) is a linear function and \( f(t_1) = (-2, -1) \) and \( f(t_2) = (3, 2) \).

What is \( f(t_1 + t_2) \)?

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Suppose \( f(t) : \mathbb{R} \rightarrow \mathbb{R}^2 \) is a linear function and \( f(t_1) = (-2, -1) \) and \( f(t_2) = (3, 2) \).

\( f(t_2) = (4, 2) \)

What is \( f(t_1 + t_2) \)?

---

Suppose \( f(t): \mathbb{R} \rightarrow \mathbb{R}^2 \) is a linear function and \( f(t_1) = (-2, -1) \).

What is \( f(t) \)?

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\( f(1) \)

\( f(t) \) is completely determined by its value at a non-zero \( t \) (because \( f(0) = 0 \))
**Affine Functions**

\[ f(t) \text{ is affine if } f(\alpha t_1 + \beta t_2) = \alpha f(t_1) + \beta f(t_2) \text{ when } \alpha + \beta = 1 \]

\[ f(t) \text{ is completely determined by its value at two points} \]

`f(t)` is a function in \( \mathbb{R} \rightarrow \mathbb{R}^2 \) such that \( f(0) = (-1, -1) \) and \( f(1) = (4, 2) \).

What is \( f(0.5t_1 + 0.5t_2) \)?

**Multi-Affine Functions**

\[ f(u_1, u_2, \ldots, u_n) \text{ is multi-affine if holding all but one parameter fixed yields an affine function} \]

\[ u(t_1, t_2) \text{ is a multi-affine, symmetric function} \]

\[ u(0,1) = u(1,0) = p_1 \]

\[ u(0,0) = p_0 \]

\[ u(1,1) = p_1 \]

Where is \( u(0.5,0) \)?

Where is \( u(0.5,1) \)?

Where is \( u(0.5,0.5) \)?

**Linear & Affine**

- \( f(t) = at \) is linear
- \( f(t) = at + b \) is affine
$u(t_1, t_2)$ is a multi-affine, symmetric function

$u(0,0) = p_0$

$u(0,1) = u(1,0) = p_1$

$u(1,1) = p_1$

Where is $u(0.5,0)$?

Where is $u(0.5,1)$?

Where is $u(0.5,0.5)$?

Polar form

Given: $X(t) = at^2 + bt + c$

Construct: $x(t_1, t_2)$ such that

- $x(t, t) = X(t)$
- $x(t_1, t_2) = x(t_2, t_1)$

$x(t_1, t_2) = a t_1 t_2 + (b/2)(t_1 + t_2) + c$

$x(t_1, t_2)$ is a multi-affine function

Polar form

$x(t_1, t_2) = a t_1 t_2 + (b/2)(t_1 + t_2) + c$

Hold $t_1$ fixed:

$x'(t_2) = (at_1 + b/2)t_2 + (b/2)t_1 + c$

similarly for $t_2$

Cubic Bezier

Where is $U(1/2)$?

$U(0) = u(0,0,0)$

$U(0) = u(0,0,0)$

$U(0,0,0)$

$U(1) = u(1,1,1)$

$U(1) = u(1,1,1)$

$U(0) = u(0,0,0)$

$U(0) = u(0,0,0)$