CS 182, Advanced Algorithms
Spring 2003
Homework 1b
Due Tuesday, January 28

1. **[20 Points] Stirling Numbers and The Discrete Calculus.** Recall that in class, we defined the Stirling Numbers of the Second Kind. On the last homework assignment, you gave a combinatorial proof that:

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}$$

From this identity, we could build the Stirling Triangle of the Second Kind. Then, Ran made the following wild claim in class:

$$x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}}$$

for any integer $n \geq 0$. This identity was very useful in computing all sorts of summations using the Discrete Calculus. Now, you will prove this identity.

   (a) First, show that $x \cdot x^{\underline{k}} = x^{\underline{k+1}} + kx^{\underline{k}}$.

   (b) Now, use induction on n to show that

$$x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}}$$

   The identity that you proved in part (*a*) may be useful to you.

2. **[40 Points] Another Version of Discrete Calculus!** In this problem we explore another version of discrete calculus that is similar, but not identical, to the version we examined in class.

   (a) **[5 Points] Warmin' Up!** In class, we defined $\Delta(f(x))$ to be $f(x+1)-f(x)$. Here, we'll explore a new discrete derivative Δ' defined by $\Delta'(f(x)) = f(x) - f(x-1)$.

      i. What is $\Delta'(x^2)$?

      ii. What is $\Delta'(x^{\underline{2}})$?

   (b) **[5 Points] Yield to the Rising Power!** Aha! That didn't turn out so great. Let's apply the standard trick of "defining our way out of trouble!" In particular, we'll define yet another type of exponentiation. Let $x^{\overline{m}}$, pronounced "x to the m rising", be defined by $x(x+1)(x+2)\ldots(x+m-1)$. (Notice that this is the product of m consecutive terms.) What is $\Delta'(x^{\overline{m}})$? Show your work.

   (c) **[5 Points] Checking out the Properties of Δ'.**

i. Use the definition of Δ' to prove that $\Delta'(f(x)+h(x)) = \Delta'(f(x)) + \Delta'(h(x))$.

ii. Show that $\Delta'(c \cdot f(x)) = c \cdot \Delta'(f(x))$.

(d) [**10 Points**] **And now for the New Definite Summation!** Now we are compelled to define $\sum_a^b{}' f(x)\delta x$ to be $g(b) - g(a)$ where $\Delta'(g(x)) = f(x)$. Prove the Second Fundamental Theorem of Discrete Calculus:

$$\sum_a^b{}' f(x)\delta x = \sum_{k=a+1}^{b} f(k).$$

(e) [**5 Points**] **And now Some Applications...** Use the Second Fundamental Theorem of Discrete Calculus to find closed forms for each of the summations below. Be sure to show your work.

i. $\sum_{k=1}^{n} k$.

ii. $\sum_{k=1}^{n} k^2$.

(f) [**5 Points**] **Summation by Parts!** Now expand $\Delta'(u(x)v(x))$ and derive a new summation by parts formula from it.

(g) [**5 Points**] **Using Summation by Parts.** So isn't this slick?! Now, use the Summation by Parts formula above to find a nice closed form for

$$\sum_{k=1}^{n} k2^k.$$

Be sure to show each step of your work.

3. [**15 Points**] **The Rise and Fall of the Exponent!** In this problem we'll take one last look at the Discrete Calculus.

(a) How should we define $x^{\overline{-m}}$? Check that your definition is "good" by making sure that $\Delta'(x^{\overline{-m}}) = -mx^{\overline{-(m+1)}}$.

(b) Now we'll see that rising powers and falling powers are intimately related! First, show that $x^{\overline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$.

(c) Next, show that $x^{\underline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$.