Binary Relations and Graphs
Binary Relations

- A binary relation is a set of ordered-pairs, with the elements of each pair drawn from a common set.

- Example: \{ [1, 2], [2, 3], [1, 3] \}
Binary Relations

- (A **binary relation** is a set of ordered-pairs, with the elements of each pair drawn from a common set, called the **domain**.)

- A finite set can be implemented as a list.

- An ordered-pair can be represented as a list.

- **Therefore**, a **finite** binary relation can be represented as a list of lists of two elements each.
Consider the binary relation “can be donor for” on the domain of blood types: {“A”, “B”, “AB”, “O”}

As a list, this relation could be represented
[[“A”, “A”], [“A”, “AB”],
[“B”, “B”], [“B”, “AB”],
[“AB”, “AB”],
[“O”, “A”], [“O”, “AB”], [“O”, “B”], [“O”, “O”]]
Another Binary Relation Implementation

- With each first element of some pair, give the list of second elements to which the first element is related:
  - [[“A”, [“A”, “AB”]],
    [“B”, [“B”, “AB”]],
    [“AB”, [“AB”]],
    [“O”, [“O”, “A”, “B”, “AB”]]]

- Compare the two representations.
A Directed Graph is a way of presenting a binary relation:

- The **nodes** (shapes) of a directed graph correspond to the elements in the domain.
- The **arcs** (arrows) of a directed graph correspond to the pairs that are related.
A Small Directed Graph Example

- For the binary relation can be donor for represented as a list previously

  ```
  ["A", "A"], ["A", "AB"], ["B", "B"], ["B", "AB"], ["AB", "AB"],
  ["O", "A"], ["O", "AB"], ["O", "B"], ["O", "O"]
  ```

  the directed graph would be
Each page in the world-wide web can be considered a node.

Each hyper-link from one page to another is an arc.

cf. graphics.stanford.edu/papers/webviz/
The previous relation example illustrates two common properties that a binary relation may have:

- **Transitive property**: For every $x, y, z$ in the domain, if $x$ is related to $y$ and $y$ is related to $z$, then $x$ is related to $z$.

- **Reflexive property**: For every $x$ in the domain, $x$ is related to $x$.

* (Note that any of $x, y, z$ may be equal!)
The previous example has only the second of the following additional two common properties:

- **symmetric** property: For every \( x, y \) in the domain if \( x \) is related to \( y \) then \( y \) is related to \( x \).

- **anti-symmetric** property: For every \( x, y \) in the domain if \( x \) is related to \( y \) and \( y \) is related to \( x \), then \( x = y \).
A relation with the reflexive, anti-symmetric, and transitive properties is called a partial order.

Example:
A relation with the reflexive, symmetric, and transitive properties is called an **equivalence relation**. Such a relation generalizes the notion of equality, since in this case if $x$ is related to $z$ and $y$ is related to $z$, then $x$ is related to $y$.

In other words, if each of a set of elements is related to a common thing, the elements in the set and the common thing are all related to each other.

**Equivalence:**
Example of an Equivalence Relation

- Consider the relation "is a homophone of" ("sounds the same as") on a set of words, such as {"air", "ere", "heir", "buy", "by", "bye", "dew", "do", "due", "ewe", "you", "yew"}
  - **Reflexive**: Every x is a homophone of x.
  - **Symmetric**: If x is a homophone of y then y is a homophone of x.
  - **Transitive**: If x is a homophone of y and y is a homophone of z, then x is a homophone of z.
- Therefore this is an equivalence relation.
Example of not an Equivalence Relation

- Consider the relation “is a synonym for” ("means the same as")
  - Reflexive: Every $x$ is a synonym for $x$.
  - Symmetric: If $x$ is a synonym of $y$ then $y$ is a synonym of $x$.
  - Transitive: NOT: If $x$ is a synonym of $y$ and $y$ is a synonym of $z$, then $x$ is a synonym of $z$.
    (Example: “angry” is a synonym for “mad”, and “mad” is a synonym for “insane”, but “angry” is not a synonym for “insane”.)
- Therefore this is a not an equivalence relation.
An undirected graph is a way of presenting a symmetric binary relation:

Since whenever x is related to y also y is related to x, we don’t have to show direction with arcs. Instead of calling them arcs then, it is common to call them edges.
An example of a symmetric relation and its undirected graph is “can be a synonym of“:

- "angry"
- "mad"
- "insane"
- "crazy"
- "comical"
- "cool"
- "chill"
- "relax"
- "unwind"
- "refrigerate"
- "lessen"
More Information Structures?

- There is a lot more to be said, and our next topic in this thread will be trees.

- But for now, we will discuss some ways to work with these representations in an actual language.