Functional Programming

Functional Programming

- Functional programming is one of the major **fundamental** programming paradigms.
- It means programming only by composing functions, not using assignment statements.
- It can be used in conjunction with other paradigms, such as object-oriented programming.

Functional Programming is "Complete"

- There is a certain well-defined sense in which a programming language can be called "complete":
  - The language is capable of representing *any* computable function.
  - Most languages of significance, including most functional ones, are complete in this sense.
- More on the definition of "computable" and "complete" later.

A Functional Programming Language

- We will use the language rex to exemplify functional programming.
- rex is interactive:
  - Definitions are entered.
  - Expressions are evaluated to get results.
- You may run rex on turing:
  - `turing > rex`
  - `rex >`

UNIX prompt

rex prompt

rex usage examples (user input is shown in bold)

```
rex > length([ [1, 2], [3, 4], [5, 6] ]);
3

rex > sort([3, 9, 1, 2, 8, 7, 5, 6, 4]);
[1, 2, 3, 4, 5, 6, 7, 8, 9]

rex > sort(["oats", "peas", "beans", "barley"]);
[barley, beans, oats, peas]

rex >
```

more rex usage examples
(define variables to avoid re-entry)

```
rex > x = [3, 9, 1, 2, 8, 7, 5, 6, 4];
1
```

This 1 means true, the definition was accepted.

```
rex > x;
[3, 9, 1, 2, 8, 7, 5, 6, 4]

rex > sort(x);
[1, 2, 3, 4, 5, 6, 7, 8, 9]

rex > x;
[3, 9, 1, 2, 8, 7, 5, 6, 4]
```

more rex usage examples (previous session continued)

```
rex > length(x);
9

rex > reverse(x);
[4, 6, 5, 7, 8, 2, 1, 9, 3]

rex > append(x, x);
[3, 9, 1, 2, 8, 7, 5, 6, 4,
 3, 9, 1, 2, 8, 7, 5, 6, 4]
```

Load files to prevent re-typing

contents of file text.rex, prepared with a text editor, such as Emacs:

```
// This is a set of rex definitions, with comments

// x is a list of some small random numbers.

x = [3, 9, 1, 2, 8, 7, 5, 6, 4];


// y is a list of some grains.

y = sort(["oats", "peas", "beans", "barley"]);


// z is a list of pairs

z = [ [1, 2], [3, 4], [5, 6] ];


/*
   Above you see comments to end-of-line.
   You can also have multi-line comments such as this one,
       just like Java or C++.
*/
```

At least two ways to load a file:

Method 1: Include the file name on the UNIX command line:

```
unix > rex test.rex
test.rex loaded
rex > x;
[3, 9, 1, 2, 8, 7, 5, 6, 4]

rex > y;
[barley, beans, oats, peas]

rex > z;
[[1, 2], [3, 4], [5, 6]]
```

here

You can re-run the command without retyping, e.g. `unix > !r`

At least two ways to load a file:

Method 2: Include the file from a rex command line

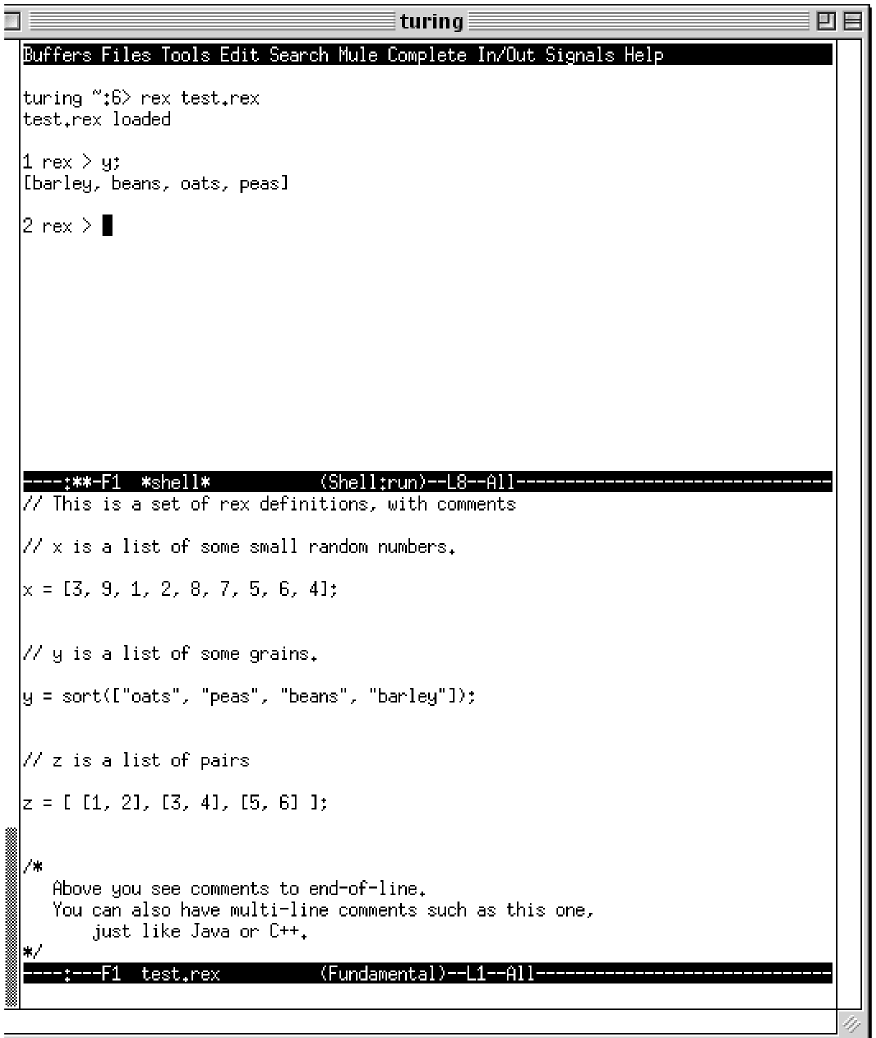
```
unix > rex
rex > *i test.rex
read file test.rex
rex > x;
[3, 9, 1, 2, 8, 7, 5, 6, 4]

rex > y;
[barley, beans, oats, peas]

rex > z;
[[1, 2], [3, 4], [5, 6]]
```

here

Split-screen editing in Emacs
(what I use most of the time)



UNIX shell in emacs window

In Emacs:

`control-x 2` to split window

`escape-x shell` to get shell

Can cut/paste using only keystrokes

your rex file for editing

High-Level
Functional Programming

High-Level Functional Programming

- By ***high-level*** we mean that we are only going to construct functions by composing together (usually powerful) built-in functions.

- We place the construction of functions based on the list dichotomy, for example, under *low-level*.

Some Built-in Functions in rex

- We already saw examples:
  - **length**: returns the length of a list
  - **member**: tells whether something is in a list
  - **sort**: returns a sorted version of a list
  - **reverse**: returns the reverse of a list
  - **append**: appends together two lists
- Other functions follow

zip

- **zip** "zips together" two lists:
  - zip([3, 5, 7], [11, 13, 17]) ⇨

    [3, 11, 5, 13, 7, 17]

first

- **first** returns the first element of a non-empty list:
  - first([3, 5, 7, 11, 13]) ⇨ 3
  - first([[3, 5, 7], 11, 13]) ⇨ [3, 5, 7]
- first([ ]) doesn't make sense; it returns an **error value**
- Be sure that the argument to first is not [].

rest

- **rest** returns a list of all but the first element of a non-empty list:
  - rest([3, 5, 7, 11, 13]) ⇨ [5, 7, 11, 13]
  - rest([[3, 5, 7], 11, 13]) ⇨ [11, 13]
- rest([ ]) doesn't make sense; it returns an **error value**
- Be sure that the argument to rest is not [].

cons

- **cons** creates a list from a first element and another list:
  - cons(3, [5, 7, 11, 13]) ⇨ [3, 5, 7, 11, 13]
  - cons([3, 5, 7], [11, 13]) ⇨ [[3, 5, 7], 11, 13]

- IMPORTANT: cons is not *append*:
  - append([3, 5, 7], [11, 13]) ⇨ [3, 5, 7, 11, 13]

Type Signature

- Suppose T is some data type
- Let T* mean the type of lists of elements of type T. Here are some **type signatures**:
- cons: T □ T* □ T*
- append: T* □ T* □ T*
- first: T* □ T
- rest: T* □ T*
- Here □ means the *pairing* of arguments.

range

- **range** produces a "range" of numbers
- range(1, 10) ⇨ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

- There is also a 3-argument version, in which the increment can be specified:
- range(1, 4.5, 0.5) ⇨ [1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5]
- Type signature of range?

scale

- **scale** multiplies the values in a list by a common factor
- scale(3, [2, 4, 6, 8]) ⇨ [6, 12, 18, 24]
- Type signature of scale?

assoc

- **assoc** “looks up” a value in an association list.
  - If found, the entire pair is returned.
  - If not found, [ ] is returned.
- assoc(“c”, [[“a”, 3], [“b”, 5], [“c”, 7]]) ➪ [“c”, 7]
- assoc(“d”, [[“a”, 3], [“b”, 5], [“c”, 7]]) ➪ [ ]
- Type signature of assoc?

remove_duplicates

- **remove_duplicates** returns a new list with the 2nd, 3rd, … of any element removed
- remove_duplicates([2, 3, 4, 5, 2, 6, 5, 4]) ➪ [2, 3, 4, 5, 6]

Predicates

- A ***predicate*** is a function that returns one of two values, for purposes of discrimination among arguments.
- In rex, the two values of interest are:
  - 1, for true
  - 0, for false
- Some built-in rex predicates follow

null predicate

- null tests a list for being empty:
  - null([ ]) ➪ 1
  - null([1]) ➪ 0
- Type signature of null?

member predicate

- **member**(X, L) tells whether or not X occurs in list L
- member(11, [5, 7, 11, 13]) ⇨ 1
- member(12, [5, 7, 11, 13]) ⇨ 0

even predicate

- **even**(X) tells whether or not X is evenly divisible by 2.
- even(11) ➪ 0
- even(12) ➪ 1
- Note: The argument must be an integer.

odd predicate

- **odd**(X) tells whether or not X divided by 2 has a remainder of 1.
- odd(11) ⇨ 1
- odd(12) ⇨ 0
- Note: The argument must be an integer.

is_prime predicate

- **is_prime**(X) tells whether or not X is prime (has any even divisors other than itself and 1)
- is_prime(11) ⇨ 1
- is_prime(12) ⇨ 0
- Note: The argument must be an integer.

“satisfy”

- When an argument value makes a predicate return value 1 (true), the argument is said to **satisfy** the predicate.

- This is useful in constructing sentences where the argument to the predicate is treated as active and the predicate is passive.

"satisfy" Example

- The predicate is_prime is satisfied by each of 2, 3, 5, 7, 11, …
- It is not satisfied by 4, 6, 8, 9, 10, …

Higher-Order Functions

- By a higher-order function, we mean one that either:
  - takes a function as an argument, or
  - returns a function as a value
- Predicates are special cases of functions.

map

- **map** is an extremely useful function.
- Its first argument is a function of one argument.
- Its second argument is a list of values of the same type as the argument to the first argument.
- It applies the first argument to all of the elements in the list, giving a list as the result.

map Examples

- map(odd, [2, 3, 4, 5, 6, 7, 8, 9])
  ➪ [0, 1, 0, 1, 0, 1, 0, 1]

- map(is_prime, [2, 3, 4, 5, 6, 7, 8, 9])
  ➪ [1, 1, 0, 1, 0, 1, 0, 0]

- square(X) = X*X;

  map(square, [2, 3, 4, 5, 6, 7, 8, 9])
  ➪ [4, 9, 16, 25, 36, 49, 64, 81]

In rex, we can define functions by equations this way.

Exercise

- Give a type signature for map.
- (Hint: Let T stand for the type of elements in the list.)

3-argument map in rex

- This version of map is defined similarly, but
  - The first argument is a binary (2-argument) function;
  - The 2nd and 3rd arguments are both lists.
- The function argument is applied to pairs of corresponding elements, one from each list.

3-argument map

- map(F, [x_1, x_2, x_3, …, x_n], [y_1, y_2, y_3, …, y_n]) ⇨ [F(x_1, y_1), F(x_2, y_2), …, F(x_n, y_n)]

- Examples:
  - map(+, [1, 2, 3], [4, 5, 6]) ⇨ [5, 7, 9]
  - map(*, [1, 2, 3], [4, 5, 6]) ⇨ [4, 10, 18]
  - map(list, [1, 2, 3], [4, 5, 6]) ⇨ [[1, 4], [2, 5], [3, 6]]

Exercise

- Give a type signature for the 3-argument map.
- (Note: The lists don't have to have the same type of element as each other.)

keep

- **keep** has a first argument that is a predicate and a second argument that is a list.
- It returns the list of values that satisfy the first argument.
- keep(odd, [3, 4, 6, 5, 11, 12, 22, 31])
  ⇨ [3, 5, 11, 31]

drop

- **drop** is like *keep*, except that it returns the list of values that do not satisfy the predicate argument.
- drop(odd, [3, 4, 6, 5, 11, 12, 22, 31])
  ⇨ [6, 12, 22]
- is_zero(X) = X == 0;
  drop(is_zero, [4, 6, 2, 0, 1, -5, 0])
  ⇨ [4, 6, 2, 1, -5]

Exercise

- *keep* and *drop* both have the same type signature; what is it?

reduce

- ***reduce*** takes three arguments:
  - a binary operator, say b, of type V □ V □ V;
    b should be associative: b(x, b(y, z)) = b(b(x, y), z)
  - a value u of type V
  - a list L = [x_1, x_2, x_3, …, x_n] of values of type V
- It returns a single value of type V:
  - If L is empty, then the value returned is u.
  - If L is not empty, the value is

$$b(\ldots b(b(b(u, x_1), x_2), x_3), \ldots, x_n)$$

Units

- If the first argument of reduce is an algebraic operator, then
- Normally the second argument is the ***unit*** for that operator.
- A unit has the property that for any X,

  $$b(u, X) = b(X, u) = X.$$
- 0 is the unit for +, 1 is the unit for *,
  [] is the unit for append.

Exercise

- What is an appropriate unit for:
  - max
  - min

reduce Examples

- reduce(+, 0, [6, 7, 8, 9]) ➪ 30
- reduce(*, 1, [6, 7, 8, 9]) ➪ 3024
- reduce(append, [ ], [[1, 2, 3], [4, 5], [6]])
  ➪ [1, 2, 3, 4, 5, 6]

Anonymous Functions

- Sometimes it may be regarded as inconvenient to **name** functions such as isZero.
- Another problem arises when we want to **fix** one or more arguments to a function, leaving the remainder to vary.
- Both are solved by *anonymous* functions.

Anonymous Functions

- Functions have a **meaning** independent of the **names** we give them.
- We want a way to use a function without giving it a name.
- Notation:

  (X) => … some expression …

  means "the function that, with argument X, returns the value of … some expression ….

Example

- The function isZero, defined by:

  isZero(X) = X == 0;

  can also be written *anonymously*:

  (X) => X == 0

  read "the function that, with argument X, returns the value of X == 0".

Precedent

- This notation for talking about a function goes back to (at least) Bourbaki (French Mathematics Group), where the symbol

  $$\mapsto$$

  was used instead of

  $$=>$$

- Alonzo Church used the idea extensively, but with a different symbol □ as a *prefix*. This notation requires some acclimation.

More Anonymous Functions

- (X) => X+5 — The function that adds 5
- (X) => X*5 — The function that multiplies by 5
- (X) => X*X — The function that squares
- (X, Y) => Y/X — The function that divides its second argument by its first.

Sample Usage

- map((X)=>X+5, [1, 2, 3, 4])

  ⇨ [6, 7, 8, 9]

- map((X)=>X*X, [1, 2, 3, 4])

  ⇨ [1, 4, 9, 16]

Exercise

- Give an equation defining *scale* using *map*, where
  scale(F, L) multiplies each element of L by a factor F.

Anonymous Functions with "Imported" Values

- drop_multiples(X, L) =
  drop((Y) => (Y%X == 0), L)

  The predicate that tests divisibility by X.

- Here X is **imported** to the anonymous function; it is not an argument to it.

- This form of usage is **VERY IMPORTANT**.

Exercises

- Give an equation defining pairWith, such that

  pairWith(X, L) creates a list in which each element of L is paired with X:

  pairWith(3, [1, 2, 3])

  ⇨ [[3, 1], [3, 2], [3, 3]]

Exercises

- Can you give an equation defining a function **pairs**, such that

    pairs(L, M) creates a list in which each element of L is paired with each element of M, e.g.

    pairs([1, 2, 3], [4, 5, 6])

    ⇨ [[1, 4], [1, 5], [1, 6],
    [2, 4], [2, 5], [2, 6],
    [3, 4], [3, 5], [3, 6]]

find function

- find(P, L) returns the longest suffix of L that begins with an element satisfying P.
- Example:
  - find(odd, [2, 4, 6, 7, 9, 10, 12])
    ⇨ [7, 9, 10, 12]
- As with map, etc., find is often used with anonymous functions.

find_index function

- find_index(P, L) returns the index of the first element L that begins with an element satisfying P.
- Example:
  - find_index(odd, [2, 4, 6, 7, 9, 10, 12])
    ⇨ 3
- Indices start with 0 as for the first element of the list.

find_indices function

- find_indices(P, L) returns the list of indices of elements of L that satisfy P.
- Example:

  find_indices(odd, [2, 4, 6, 7, 9, 8, 12, 13])

  ➪ [3, 4, 7]

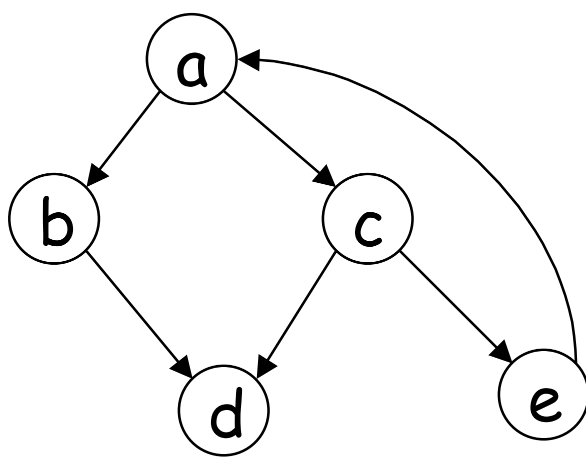
Function Decomposition

- This means: Implement a complex function in terms of simpler ones.
- This idea is of universal importance.
- Those simpler functions can be implemented in terms of still-simpler ones, and so on, until we get down to built-in functions.

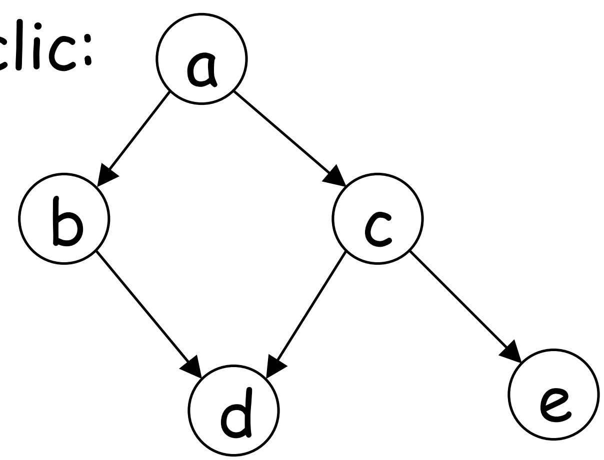
Function Decomposition Example

- Construct a function that will tell whether a directed graph, represented as a list of arcs, is acyclic.

cyclic:



acyclic:

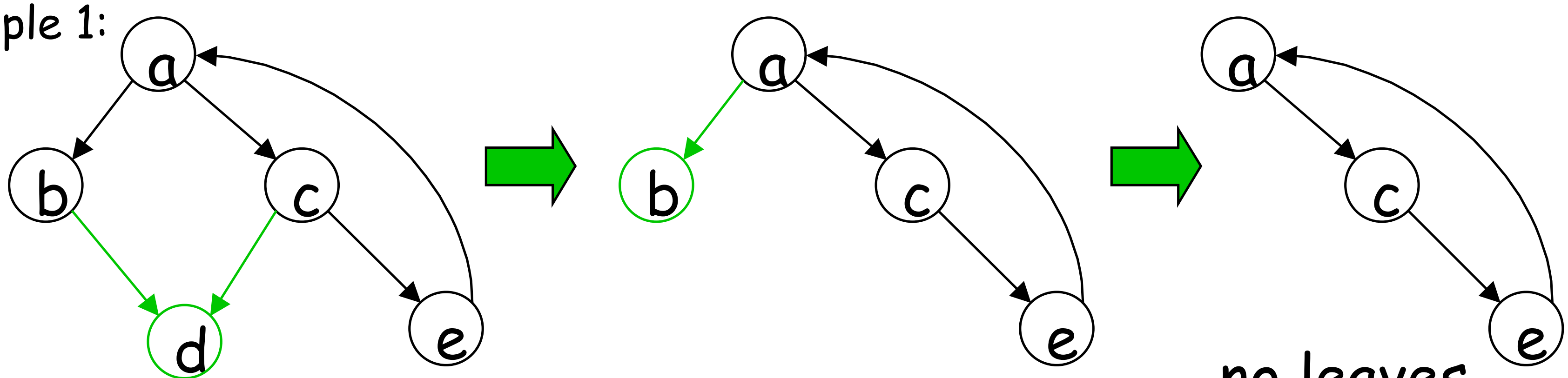


"Pruning" Method

- Rosalind B. Marimont
  A new method of checking the consistency of precedence matrices
  Journal of the ACM **6**, 164-171, 1959
- Pruning away any arcs that point to leaves does not change the cyclic/acyclic nature of the graph.
- Pruning such arcs may produce additional leaves.
- Prune until no further pruning is possible:
  - If the result is empty, the original graph was acyclic.
  - If not, it was cyclic.

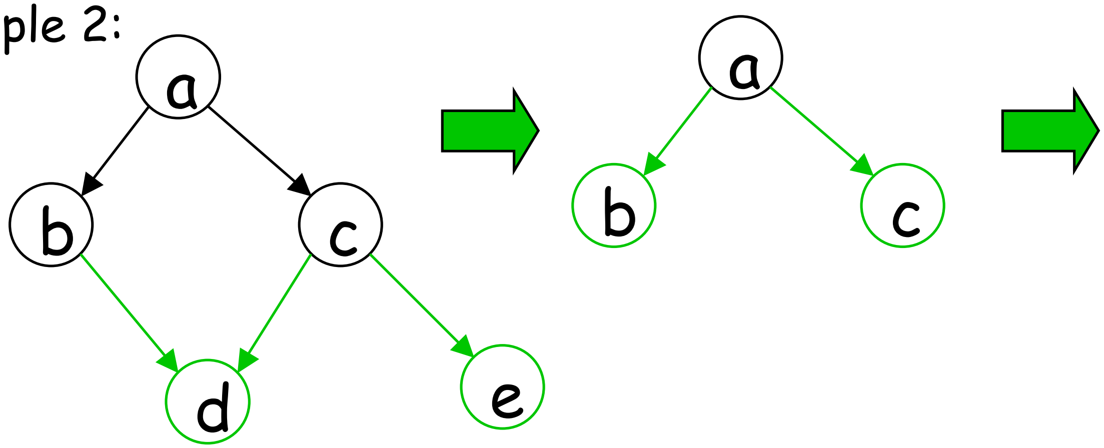
Examples of Pruning
(leaves shown in green)

example 1:



no leaves

example 2:



empty
(no arcs)

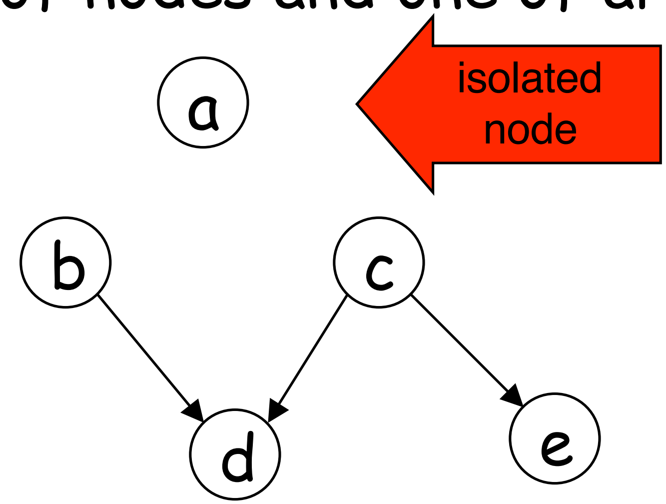
Pruning with Graphs as Lists

- Example 1:
  - [[a, b], [a, c], [b, d], [c, d], [c, e], [e, a]] ➡
  - [[a, b], [a, c], [c, e], [e, a]] ➡
  - [[a, c], [c, e], [e, a]] (no leaves)

- Example 2:
  - [[a, b], [a, c], [b, d], [c, d], [c, e]] ➡
  - [[a, b], [a, c]] ➡
  - [ ]

Note

- We are assuming that every node in the graph is on one or the other end of an arc, i.e. there are no isolated nodes, as in the graph below.
- Otherwise, we'd have to represent the graph with two lists: one of nodes and one of arcs.



Functional Code

- Basic idea:
  - As long as there is a leaf:
    Remove leaves and their attached arcs
- Translation:
- isAcyclic(Graph) =

$$\underbrace{\text{null}}_{\boxed{\text{test for empty list}}}(\underbrace{\text{iterate}}_{\boxed{\text{iterate first arg. as long as second arg. true}}}(\underbrace{\text{removeLeaves}}_{\boxed{\text{remove all leaves and their arcs}}},\ \underbrace{\text{hasLeaf}}_{\boxed{\text{test whether there is a leaf}}},\ \underbrace{\text{Graph}}_{\boxed{\text{starting graph}}}));$$

hasLeaf

- A Graph has a leaf iff isLeaf is true for one of its nodes.
- hasLeaf(Graph) =

  some((Node)=>isLeaf(Node, Graph), nodes(Graph));

test whether first arg. is true for some element of second arg.

true when Node is a leaf of this Graph

list of nodes of Graph

isLeaf

- A node is a leaf if it is not the first of any arc in the graph.
- isLeaf(Node, Graph) =

$$\underbrace{!}_{\text{‘not’ operator}}\text{member(Node, }\underbrace{\text{map(first, Graph)}}_{\text{all nodes of Graph that begin some arc}}\text{);}$$

nodes(Graph)

- nodes(Graph) =

```
        remove_duplicates(append(map(first, Graph),
                                 map(second, Graph)));
```

- remembering our assumption: that every node in the graph is on one or the other end of an arc, i.e. there are no isolated nodes, as in the graph below.

remove_leaves

- To remove the leaves:

  remove any arc that points to a leaf

- removeLeaves(Graph) =

$$\texttt{drop((Arc)=>isLeaf(}\underbrace{\texttt{second(Arc)}}_{\text{the node to which Arc points}}\texttt{, Graph),}$$

$$\underbrace{\texttt{Graph}}_{\text{the list of arcs in the graph}}\texttt{);}$$

iterate

- iterate(action, continue, State) =
  continue(State) ?
  iterate(action, continue, action(State))
  : State;

conditional expression (as in C++, Java)

P ? A : B

means if P is true then the value of the expression is A; otherwise it is B.