



Signal & Weight Vector Spaces



Vectors in \mathfrak{R}^n .

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Generalized Vectors.

X



1. An operation called vector addition is defined such that if $x \in X$ and $y \in X$ then $x+y \in X$.
2. $x+y = y+x$
3. $(x+y) + z = x + (y+z)$
4. There is a unique vector $0 \in X$, called the zero vector, such that $x+0 = x$ for all $x \in X$.
5. For each vector there is a unique vector in X , to be called $(-x)$, such that $x+(-x) = 0$.



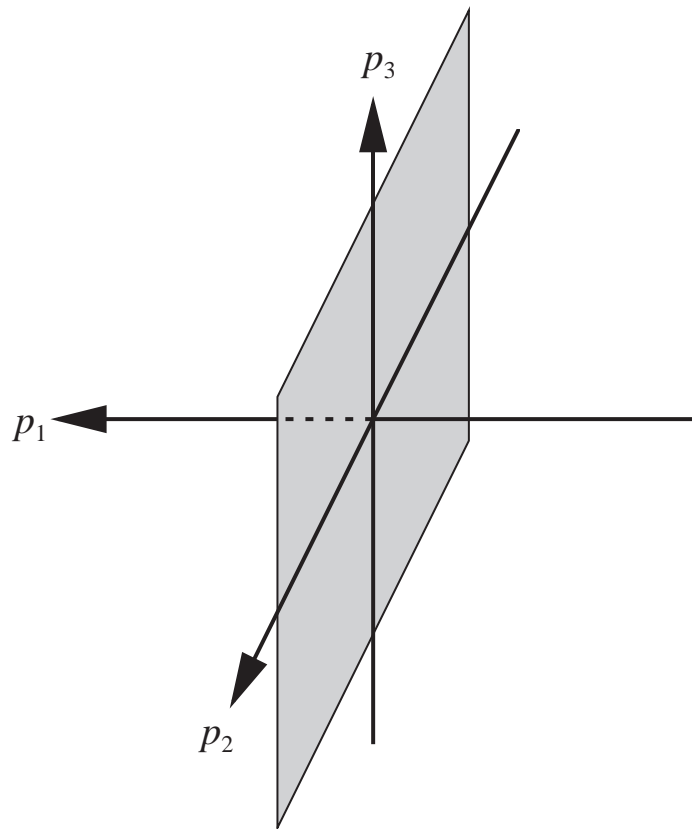
6. An operation, called multiplication, is defined such that for all scalars $a \in F$, and all vectors $x \in X$, $a x \in X$.
7. For any $x \in X$, $1x = x$ (for scalar 1).
8. For any two scalars $a \in F$ and $b \in F$, and any $x \in X$,
 $a(bx) = (ab)x$.
9. $(a+b)x = ax + bx$.
10. $a(x+y) = ax + ay$

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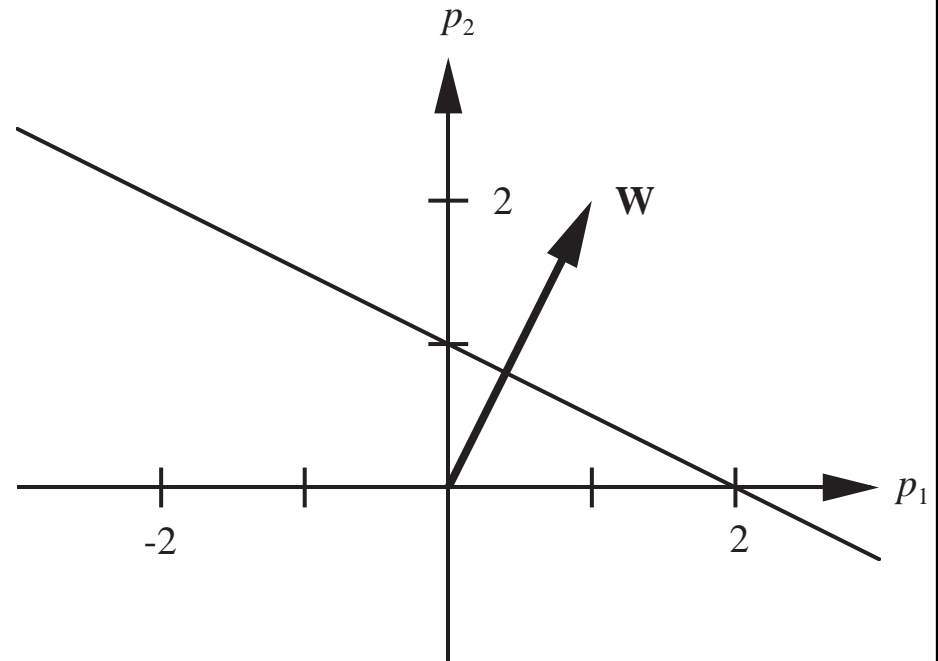
Examples (Decision Boundaries)



Is the p_2, p_3 plane a vector space?



Is the line $p_1 + 2p_2 - 2 = 0$ a vector space?



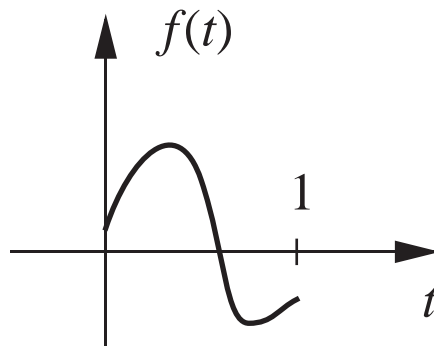


Polynomials of degree 2 or less.

$$x = 2 + t + 4t^2$$

$$y = 1 + 5t$$

Continuous functions in the interval $[0,1]$.





If

$$a_1 \mathbf{X}_1 + a_2 \mathbf{X}_2 + \cdots + a_n \mathbf{X}_n = \mathbf{0}$$

implies that each

$$a_i = 0$$

then

$$\{\mathbf{X}_i\}$$

is a set of linearly independent vectors.

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Example (Banana and Apple)



$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Let

$$a_1 \mathbf{p}_1 + a_2 \mathbf{p}_2 = \mathbf{0}$$

$$\begin{bmatrix} -a_1 + a_2 \\ a_1 + a_2 \\ -a_1 + (-a_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can only be true if

$$a_1 = a_2 = 0$$

Therefore the vectors are independent.



A subset **spans** a space if every vector in the space can be written as a linear combination of the vectors in the subspace.

$$X = x_1 \mathbf{u}_1 + x_2 \mathbf{u}_2 + \cdots + x_m \mathbf{u}_m$$



- A set of basis vectors for the space X is a set of vectors which spans X and is linearly independent.
- The dimension of a vector space, $\text{Dim}(X)$, is equal to the number of vectors in the basis set.
- Let X be a finite dimensional vector space, then every basis set of X has the same number of elements.



Polynomials of degree 2 or less.

Basis A:

$$u_1 = 1 \quad u_2 = t \quad u_3 = t^2$$

Basis B:

$$u_1 = 1 - t \quad u_2 = 1 + t \quad u_3 = 1 + t + t^2$$

(Any three linearly independent vectors
in the space will work.)

How can you represent the vector $x = 1 + 2t$ using both basis sets?



A scalar function of vectors x and y can be defined as an **inner product**, (x,y) , provided the following are satisfied (for real inner products):

- $(x,y) = (y,x)$.
- $(x,ay_1+by_2) = a(x,y_1) + b(x,y_2)$.
- $(x,x) \geq 0$, where equality holds iff $x = \theta$.

A scalar function of a vector x is called a **norm**, $\|x\|$, provided the following are satisfied:

- $\|x\| \geq 0$.
- $\|x\| = 0$ iff $x = \theta$.
- $\|a x\| = |a| \|x\|$ for scalar a .
- $\|x + y\| \leq \|x\| + \|y\|$.



Standard Euclidean Inner Product

$$\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Standard Euclidean Norm

$$\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2}$$

$$\|\mathbf{x}\| = (\mathbf{x}^T \mathbf{x})^{1/2} = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

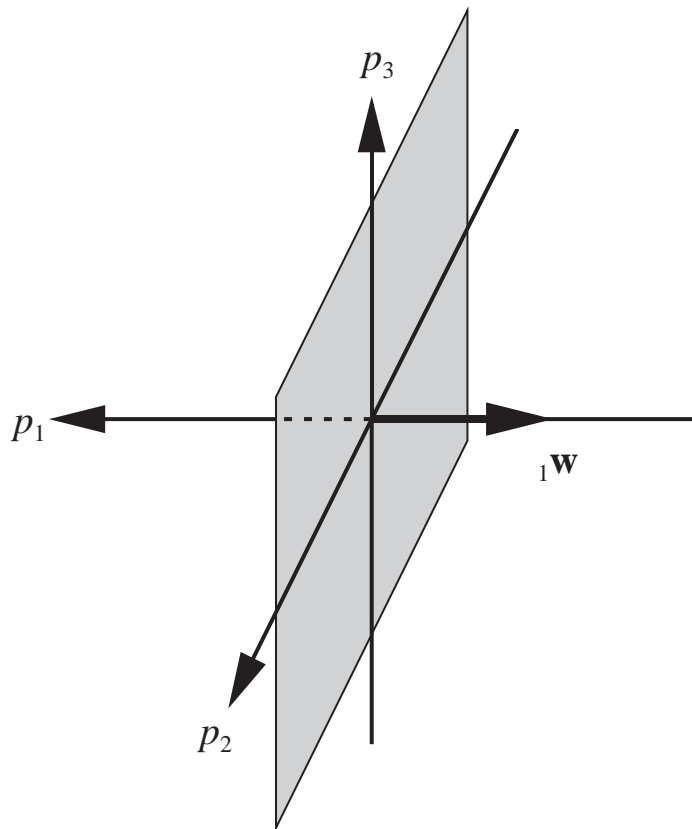
Angle

$$\cos(\theta) = (\mathbf{x}, \mathbf{y}) / (\|\mathbf{x}\| \|\mathbf{y}\|)$$



Two vectors $x, y \in X$ are orthogonal if $(x, y) = 0$.

Example



Any vector in the p_2, p_3 plane is orthogonal to the weight vector.

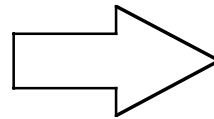
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Gram-Schmidt Orthogonalization



Independent Vectors

$$y_1, y_2, \dots, y_n$$



Orthogonal Vectors

$$V_1, V_2, \dots, V_n$$

Step 1: Set first orthogonal vector to first independent vector.

$$V_1 = y_1$$

Step 2: Subtract the portion of y_2 that is in the direction of v_1 .

$$V_2 = y_2 - aV_1$$

Where a is chosen so that v_2 is orthogonal to v_1 :

$$(V_1, V_2) = (V_1, y_2 - aV_1) = (V_1, y_2) - a(V_1, V_1) = 0$$

$$a = \frac{(V_1, y_2)}{(V_1, V_1)}$$



Projection of y_2 on v_1 :

$$\frac{(v_1, y_2)}{(v_1, v_1)} v_1$$

Step k : Subtract the portion of y_k that is in the direction of all previous v_i .

$$v_k = y_k - \sum_{i=1}^{k-1} \frac{(v_i, y_k)}{(v_i, v_i)} v_i$$

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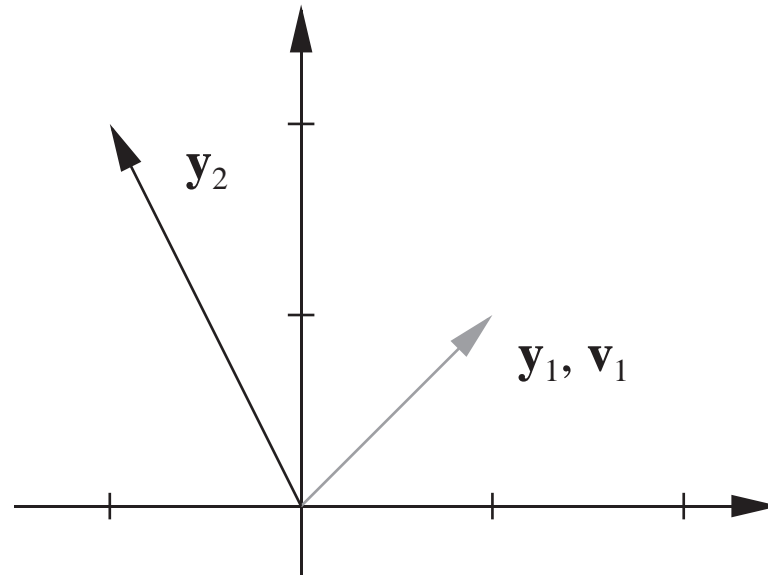
Example



$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

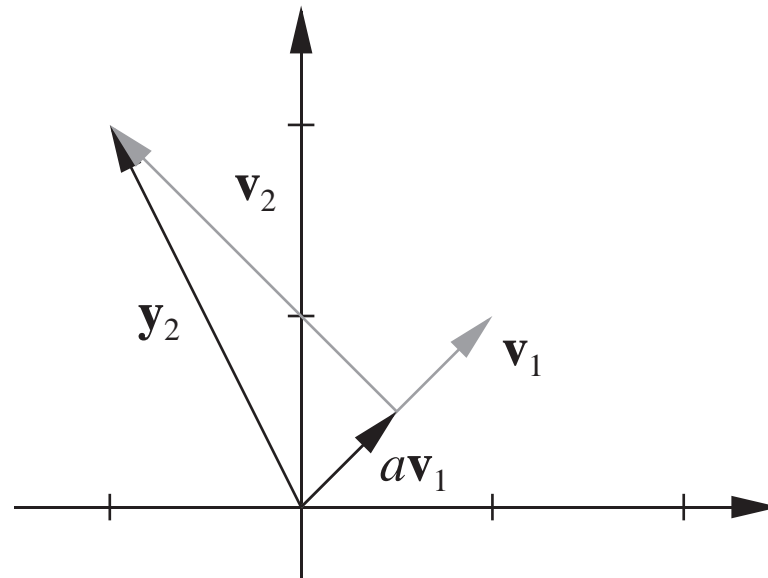
Step 1. $\mathbf{v}_1 = \mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$





Step 2.

$$\mathbf{v}_2 = \mathbf{y}_2 - \frac{\mathbf{v}_1^T \mathbf{y}_2}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 1.5 \end{bmatrix}$$





If a vector space X has a basis set $\{v_1, v_2, \dots, v_n\}$, then any $x \in X$ has a unique vector expansion:

$$X = \sum_{i=1}^n x_i v_i = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

If the basis vectors are **orthogonal**, and we take the inner product of v_j and x :

$$(v_j, X) = (v_j, \sum_{i=1}^n x_i v_i) = \sum_{i=1}^n x_i (v_j, v_i) = x_j (v_j, v_j)$$

Therefore the coefficients of the expansion can be computed:

$$x_j = \frac{(v_j, X)}{(v_j, v_j)}$$



The vector expansion provides a meaning for writing a vector as a column of numbers.

$$\mathbf{X} = \sum_{i=1}^n x_i \mathbf{V}_i = x_1 \mathbf{V}_1 + x_2 \mathbf{V}_2 + \cdots + x_n \mathbf{V}_n$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

To interpret \mathbf{x} , we need to know what basis was used for the expansion.



Definition of reciprocal basis vectors, r_i :

$$\begin{aligned} (\mathbf{r}_i, \mathbf{v}_j) &= 0 & i \neq j \\ &= 1 & i = j \end{aligned}$$

where the basis vectors are $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, and the reciprocal basis vectors are $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n\}$.

For vectors in \mathfrak{R}^n we can use the following inner product:

$$(\mathbf{r}_i, \mathbf{v}_j) = \mathbf{r}_i^T \mathbf{v}_j$$

Therefore, the equations for the reciprocal basis vectors become:

$$\mathbf{R}^T \mathbf{B} = \mathbf{I} \quad \Rightarrow \quad \mathbf{R}^T = \mathbf{B}^{-1}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \dots & \mathbf{r}_n \end{bmatrix}$$



$$\mathbf{X} = x_1 \mathbf{V}_1 + x_2 \mathbf{V}_2 + \cdots + x_n \mathbf{V}_n$$

Take the inner product of the first reciprocal basis vector with the vector to be expanded:

$$(\mathbf{r}_1, \mathbf{X}) = x_1 (\mathbf{r}_1, \mathbf{V}_1) + x_2 (\mathbf{r}_1, \mathbf{V}_2) + \cdots + x_n (\mathbf{r}_1, \mathbf{V}_n)$$

By definition of the reciprocal basis vectors:

$$(\mathbf{r}_1, \mathbf{V}_2) = (\mathbf{r}_1, \mathbf{V}_3) = \cdots = (\mathbf{r}_1, \mathbf{V}_n) = 0$$

$$(\mathbf{r}_1, \mathbf{V}_1) = 1$$

Therefore, the first coefficient in the expansion is:

$$x_1 = (\mathbf{r}_1, \mathbf{X})$$

In general, we then have (even for nonorthogonal basis vectors):

$$x_j = (\mathbf{r}_j, \mathbf{X})$$

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Example

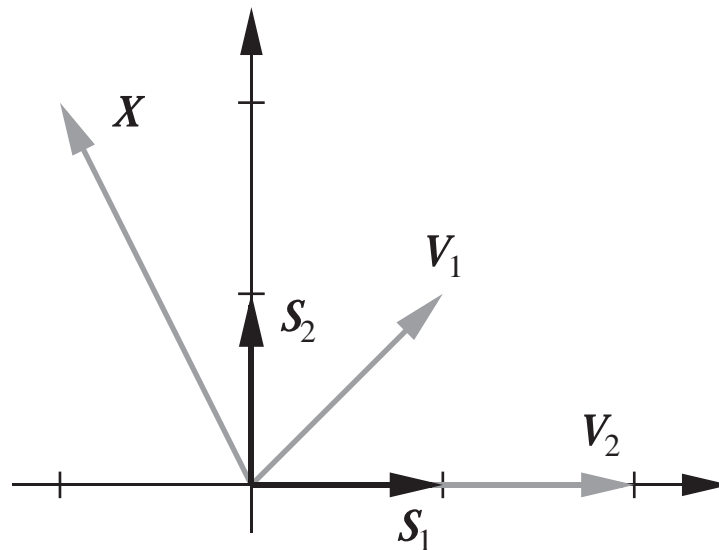


Basis Vectors:

$$\mathbf{v}_1^s = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2^s = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Vector to Expand:

$$\mathbf{x}^s = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Example (Cont.)



Reciprocal Basis Vectors:

$$\mathbf{R}^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix} \quad \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

Expansion Coefficients:

$$x_1^v = \mathbf{r}_1^T \mathbf{x}^s = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2$$

$$x_2^v = \mathbf{r}_2^T \mathbf{x}^s = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1.5$$

Matrix Form:

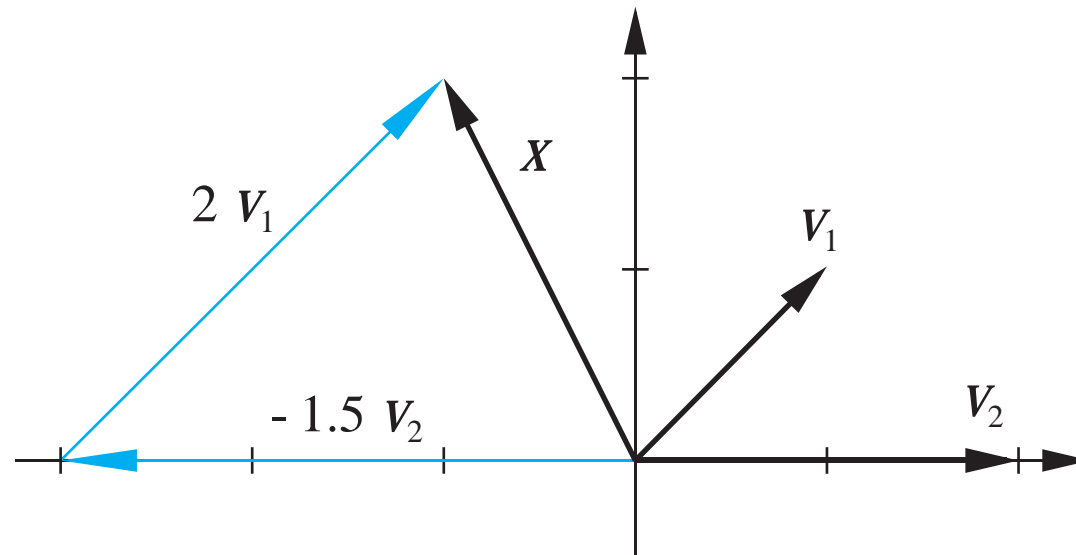
$$\mathbf{x}^v = \mathbf{R}^T \mathbf{x}^s = \mathbf{B}^{-1} \mathbf{x}^s = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}$$

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Example (Cont.)



$$\mathbf{x} = (-1)\mathbf{s}_1 + 2\mathbf{s}_2 = 2\mathbf{v}_1 - 1.5\mathbf{v}_2$$



$$\mathbf{x}^s = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\mathbf{x}^v = \begin{bmatrix} 2 \\ -1.5 \end{bmatrix}$$

The interpretation of the column of numbers depends on the basis set used for the expansion.