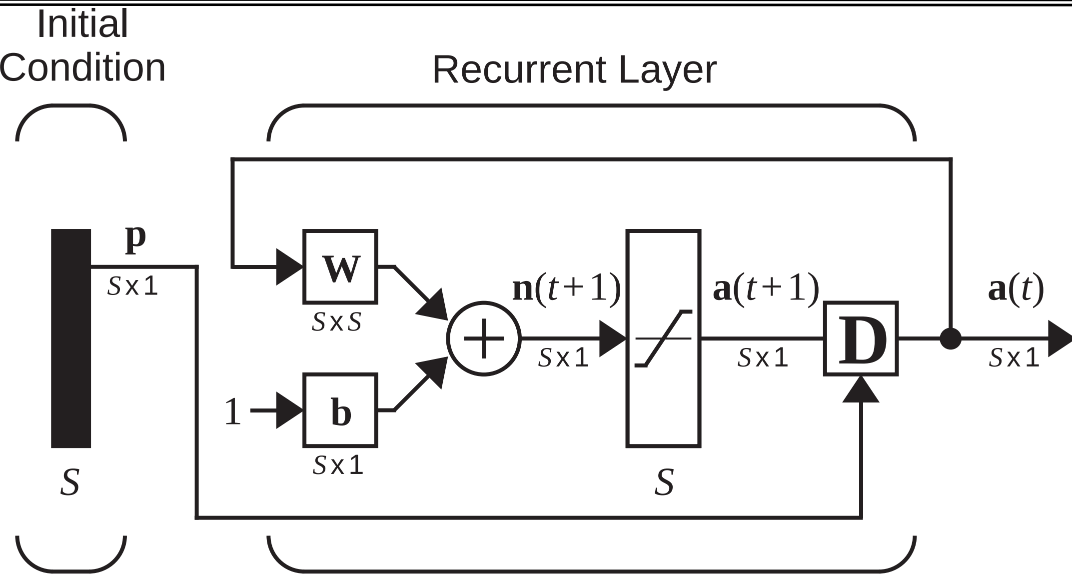
Linear Transformations

Hopfield Network Questions



$$\mathbf{a}(0) = \mathbf{p} \qquad \mathbf{a}(t+1) = \mathbf{satlins}(\mathbf{W}\mathbf{a}(t)+\mathbf{b})$$

- The network output is repeatedly multiplied by the weight matrix W.
- What is the effect of this repeated operation?
- Will the output converge, go to infinity, oscillate?
- In this chapter we want to investigate matrix multiplication, which represents a general linear transformation.

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Linear Transformations

A **transformation** consists of three parts:

1. A set of elements $X = \{x_i\}$, called the domain,
2. A set of elements $Y = \{y_i\}$, called the range, and
3. A rule relating each $x_i \in X$ to an element $y_i \in Y$.

A transformation is **linear** if:

1. For all $x_1, x_2 \in X$, $A(x_1 + x_2) = A(x_1) + A(x_2)$,
2. For all $x \in X$, $a \in \Re$, $A(ax) = aA(x)$.

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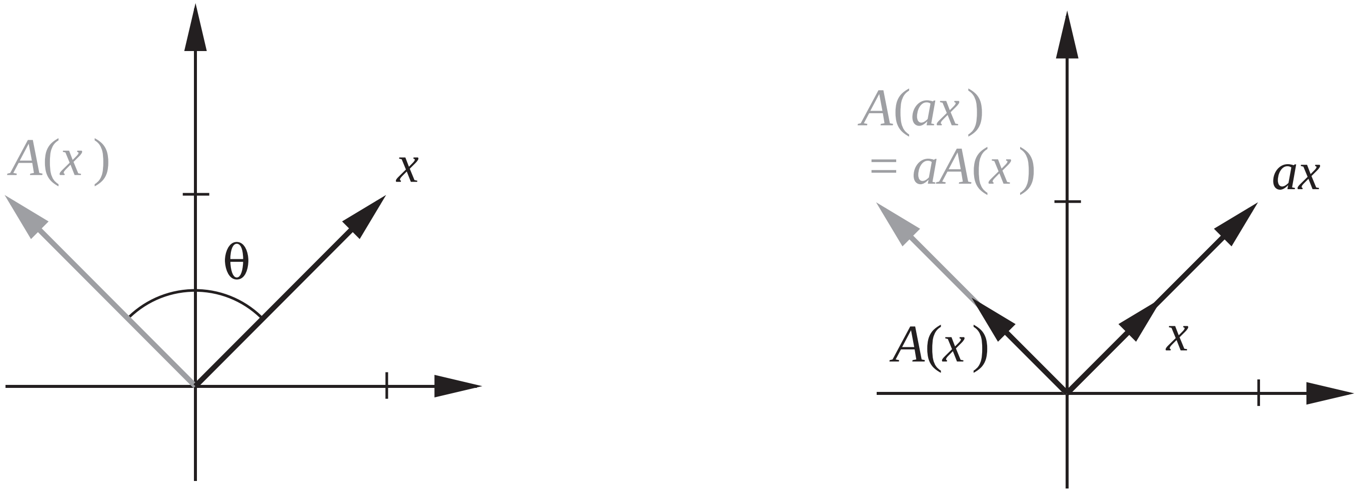
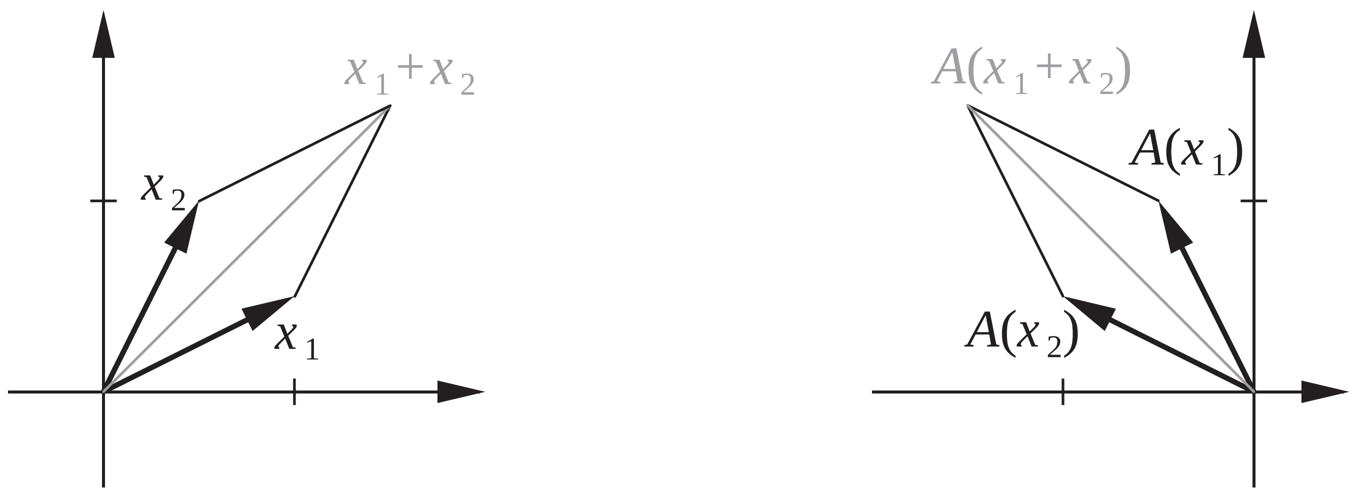
Example - Rotation

Is rotation linear?

1.



2.



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Matrix Representation - (1)

Any linear transformation between two finite-dimensional vector spaces can be represented by matrix multiplication.

Let $\{v_1, v_2, ..., v_n\}$ be a basis for X, and let $\{u_1, u_2, ..., u_m\}$ be a basis for Y.

$$x = \sum_{i=1}^{n} x_i v_i \qquad y = \sum_{i=1}^{m} y_i u_i$$

Let $A{:}X \rightarrow Y$

$$A(x) = y$$

$$A\left(\sum_{j=1}^{n} x_j v_j\right) = \sum_{i=1}^{m} y_i u_i$$

Matrix Representation - (2)

Since A is a linear operator,

$$\sum_{j=1}^{n} x_j A(v_j) = \sum_{i=1}^{m} y_i u_i$$

Since the u_i are a basis for Y,

$$A(v_j) = \sum_{i=1}^{m} a_{ij} u_i$$

(The coefficients a_{ij} will make up the matrix representation of the transformation.)

$$\sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} u_i = \sum_{i=1}^{m} y_i u_i$$

6 Matrix Representation - (3)

$$\sum_{i=1}^{m} u_i \sum_{j=1}^{n} a_{ij} x_j = \sum_{i=1}^{m} y_i u_i$$

$$\sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n} a_{ij} x_j - y_i \right) = 0$$

Because the u_i are independent,

$$\sum_{j=1}^{n} a_{ij} x_j = y_i$$

This is equivalent to matrix multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

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Summary

- A linear transformation can be represented by matrix multiplication.
- To find the matrix which represents the transformation we must transform each basis vector for the domain and then expand the result in terms of the basis vectors of the range.

$$A(v_j) = \sum_{i=1}^{m} a_{ij} u_i$$

Each of these equations gives us one column of the matrix.

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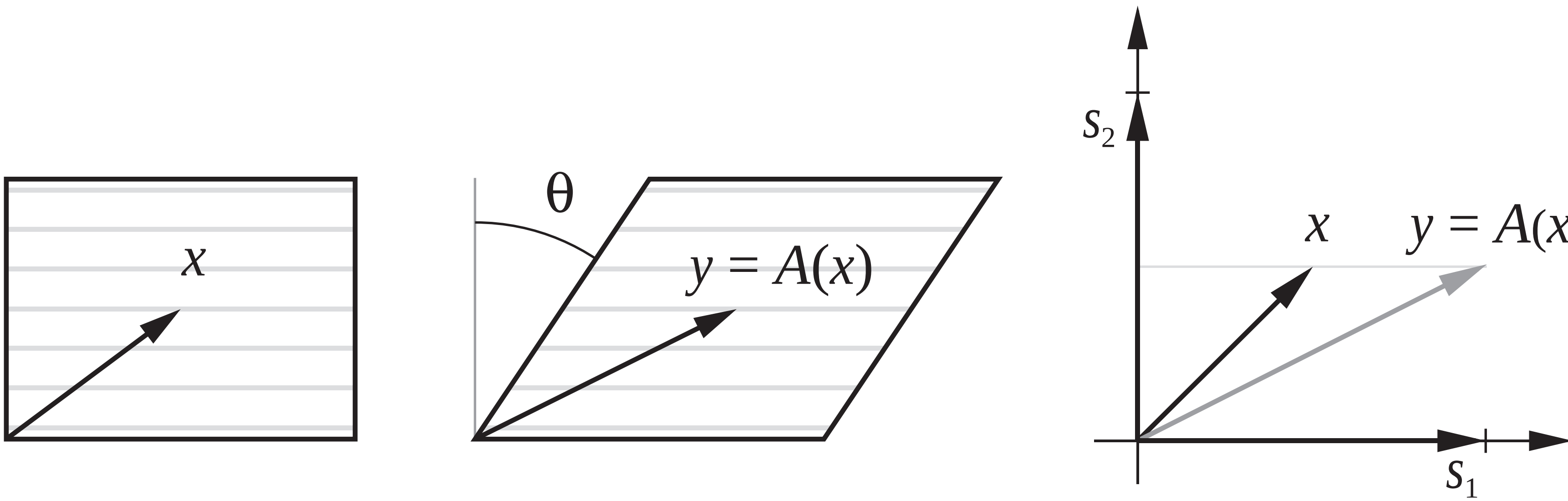
Example - (1)

Stand a deck of playing cards on edge so that you are looking at the deck sideways. Draw a vector x on the edge of the deck. Now “skew” the deck by an angle θ, as shown below, and note the new vector $y = A(x)$. What is the matrix of this transformation in terms of the standard basis set?



Example - (2)

To find the matrix we need to transform each of the basis vectors.

$$A(v_j) = \sum_{i=1}^{m} a_{ij} u_i$$

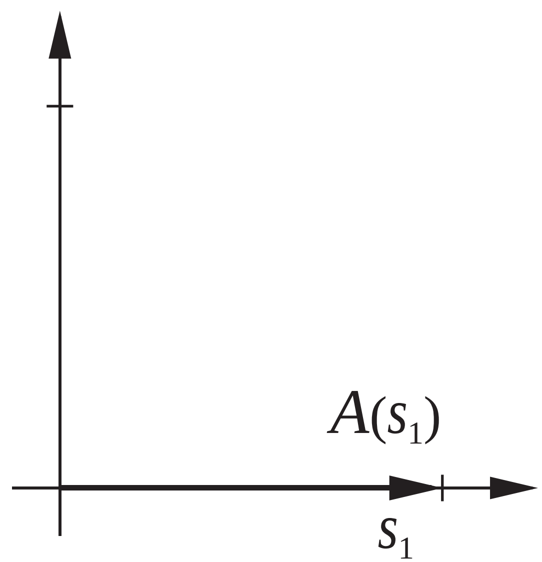
We will use the standard basis vectors for both the domain and the range.

$$A(s_j) = \sum_{i=1}^{2} a_{ij} s_i = a_{1j} s_1 + a_{2j} s_2$$

Example - (3)

We begin with s_1:

If we draw a line on the bottom card and then skew the deck, the line will not change.

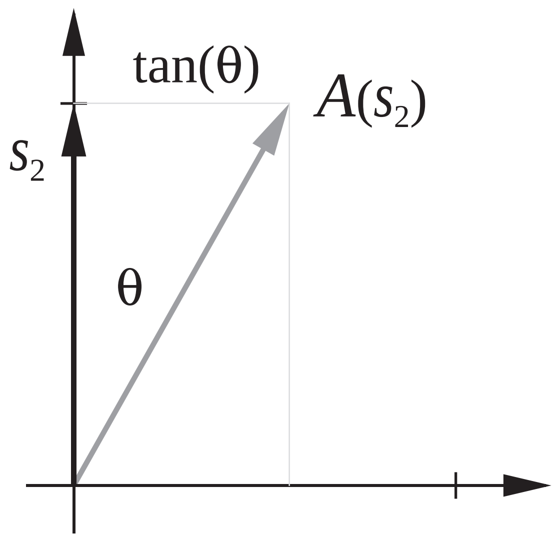


$$A(s_1) = 1s_1 + 0s_2 = \sum_{i=1}^{2} a_{i1}s_i = a_{11}s_1 + a_{21}s_2$$

This gives us the first column of the matrix.

Example - (4)

Next, we skew s_2:



$$A(s_2) = \tan(\theta)s_1 + 1s_2 = \sum_{i=1}^{2} a_{i2}s_i = a_{12}s_1 + a_{22}s_2$$

This gives us the second column of the matrix.

Example - (5)

The matrix of the transformation is:

$$\mathbf{A} = \begin{bmatrix} 1 & \tan(\theta) \\ 0 & 1 \end{bmatrix}$$

Change of Basis

Consider the linear transformation $A{:}X \rightarrow Y$. Let $\{v_1, v_2, ..., v_n\}$ be a basis for X, and let $\{u_1, u_2, ..., u_m\}$ be a basis for Y.

$$x = \sum_{i=1}^{n} x_i v_i \qquad\qquad y = \sum_{i=1}^{m} y_i u_i$$

$$A(x) = y$$

The matrix representation is:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{y}$$

New Basis Sets

Now let's consider different basis sets. Let $\{t_1, t_2, ..., t_n\}$ be a basis for X, and let $\{w_1, w_2, ..., w_m\}$ be a basis for Y.

$$x = \sum_{i=1}^{n} x'_i t_i \qquad y = \sum_{i=1}^{m} y'_i w_i$$

The new matrix representation is:

$$\begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_m \end{bmatrix}$$

$$\mathbf{A}'\mathbf{x}' = \mathbf{y}'$$

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How are **A** and **A**' related?

Expand t_i in terms of the original basis vectors for X.

$$t_i = \sum_{j=1}^{n} \mathrm{t}_{ji} v_j \qquad \mathbf{t}_i = \begin{bmatrix} t_{1i} \\ t_{2i} \\ \vdots \\ t_{ni} \end{bmatrix}$$

Expand w_i in terms of the original basis vectors for Y.

$$w_i = \sum_{j=1}^{m} w_{ji} u_j \qquad \mathbf{w}_i = \begin{bmatrix} w_{1i} \\ w_{2i} \\ \vdots \\ w_{mi} \end{bmatrix}$$

6 How are **A** and **A'** related?

$$\mathbf{B}_t = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \ldots & \mathbf{t}_n \end{bmatrix} \qquad \mathbf{x} = x'_1\mathbf{t}_1 + x'_2\mathbf{t}_2 + \cdots + x'_n\mathbf{t}_n = \mathbf{B}_t\mathbf{x}'$$

$$\mathbf{B}_w = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_m \end{bmatrix} \qquad \mathbf{y} = \mathbf{B}_w\mathbf{y}'$$

$$\mathbf{A}\mathbf{x} = \mathbf{y} \quad \Rightarrow \quad \mathbf{A}\mathbf{B}_t\mathbf{x}' = \mathbf{B}_w\mathbf{y}'$$

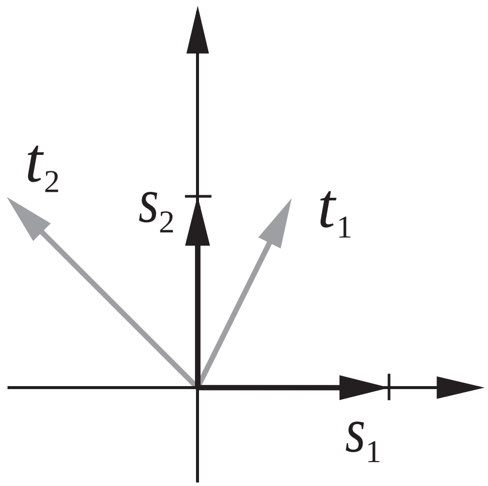
$$[\mathbf{B}_w^{-1}\mathbf{A}\mathbf{B}_t]\mathbf{x}' = \mathbf{y}'$$

$$\mathbf{A}'\mathbf{x}' = \mathbf{y}' \quad \Rightarrow \quad \mathbf{A}' = [\mathbf{B}_w^{-1}\mathbf{A}\mathbf{B}_t]$$

Similarity Transform

Example - (1)

Take the skewing problem described previously, and find the new matrix representation using the basis set $\{s_1, s_2\}$.



$$t_1 = 0.5 s_1 + s_2$$

$$t_2 = -s_1 + s_2$$

$$\mathbf{t}_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \qquad \mathbf{t}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{B}_t = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{B}_w = \mathbf{B}_t = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$

(Same basis for domain and range.)

Example - (2)

$$\mathbf{A'} = [\mathbf{B}_w^{-1}\mathbf{A}\mathbf{B}_t] = \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{A'} = \begin{bmatrix} (2/3)\tan\theta + 1 & (2/3)\tan\theta \\ (-2/3)\tan\theta & (-2/3)\tan\theta + 1 \end{bmatrix}$$

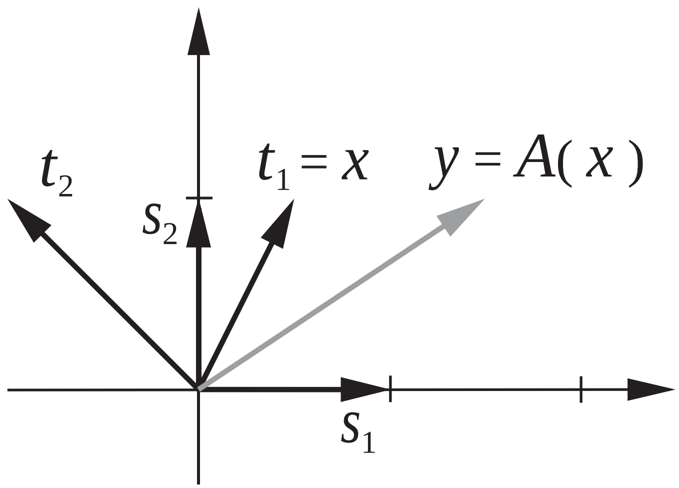
For $\theta = 45°$:

$$\mathbf{A'} = \begin{bmatrix} 5/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Example - (3)

Try a test vector: $\mathbf{x} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ $\mathbf{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} \qquad \mathbf{y}' = \mathbf{A}'\mathbf{x}' = \begin{bmatrix} 5/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$$



Check using reciprocal basis vectors:

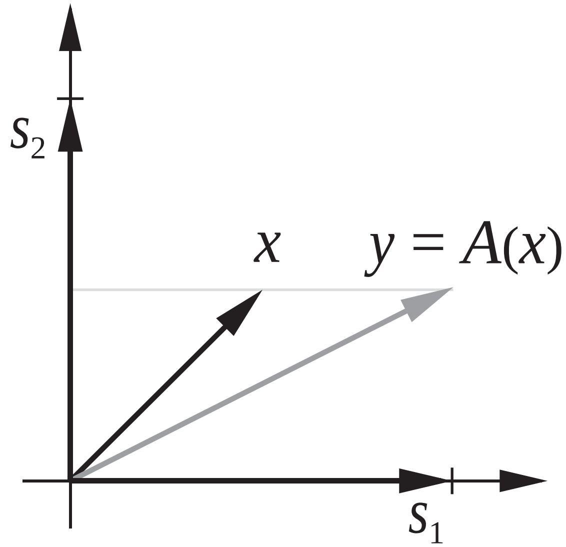
$$\mathbf{y}' = \mathbf{B}^{-1}\mathbf{y} = \begin{bmatrix} 0.5 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$$

6 Eigenvalues and Eigenvectors

Let $A:X \rightarrow X$ be a linear transformation. Those vectors $z \in X$, which are not equal to zero, and those scalars λ which satisfy

$$A(z) = \lambda z$$

are called eigenvectors and eigenvalues, respectively.



Can you find an eigenvector for this transformation?

Computing the Eigenvalues

$$\mathbf{A}\mathbf{z} = \lambda\mathbf{z}$$

$$[\mathbf{A} - \lambda\mathbf{I}]\mathbf{z} = \mathbf{0} \quad \Rightarrow \quad \left|[\mathbf{A} - \lambda\mathbf{I}]\right| = 0$$

Skewing example (45°):

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \left|\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right| = 0 \qquad (1-\lambda)^2 = 0 \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \end{matrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{z}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_{11} \\ z_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_{21} = 0 \qquad \mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For this transformation there is only one eigenvector.

6 Diagonalization

Perform a change of basis (similarity transformation) using the eigenvectors as the basis vectors. If the eigenvalues are distinct, the new matrix will be diagonal.

$$\mathbf{B} = \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \dots & \mathbf{z}_n \end{bmatrix}$$

$\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n\}$ Eigenvectors

$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ Eigenvalues

$$[\mathbf{B}^{-1}\mathbf{A}\mathbf{B}] = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

6

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right| = 0 \qquad \lambda^2 - 2\lambda = (\lambda)(\lambda - 2) = 0 \qquad \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 2 \end{matrix} \qquad \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{z}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_{21} = -z_{11} \qquad \mathbf{z}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{z}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad z_{22} = z_{12} \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Diagonal Form: $\mathbf{A}' = [\mathbf{B}^{-1}\mathbf{A}\mathbf{B}] = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$